Permutations and Combinations

Section 13.3-13.4

Dr. John Ehrke
Department of Mathematics

Fall 2012

Permutations

A **permutation** of a set is a way of counting possible arrangements of the set. In particular, a permutation answers the question,

How many ways are there to arrange items from a set in a row with no repeats?

We will introduce this concept with an example.

Example

A group of four friends are going to the movies. Only four seats are left in the theater. How many different ways are there for the friends to sit together?

Solution: If we denote the friends by the set $A = \{1, 2, 3, 4\}$. Then we can think of this as a series of four questions: (1) Who sits in the first chair? (2) Who sits in the second chair? (3) Who sits in the third chair?, and (4) Who sits in the fourth chair? There are 4, 3, 2, and 1 choices respectively for the answer to these questions. Because of the multiplication rule there are then $4 \times 3 \times 2 \times 1 = 24$ total arrangements.

A Slightly More Difficult Example

Example

Consider the same group of four friends going to the movies, but this time two of the friends are dating and want to sit next to each other. If only four seats are left in the theater, how many different ways are there for the friends to sit together?

Solution: This problem is quite a bit harder than the first. In this case there are four questions that should be asked.

1. In how many ways can the couple be seated?
2. Once the couple is seated, in how many ways can they be rearranged?
3. How can we fill in the first empty seat?
4. How can we fill in the remaining seat?

There are 3 possible seating configurations for the couple. There are 2 arrangements once the couple is seated, and there are 2 choices for the first empty seat and 1 choice for the last seat. All totaled this is $3 \times 2 \times 2 \times 1 = 12$ arrangements.

A General Permutation Formula

As we saw in the first example, the number of permutations of the four friends, was $4 \times 3 \times 2 \times 1$. In general, the number of permutations of n distinct objects is

$$n \times (n-1) \times (n-2) \cdots 2 \times 1 = n!.$$

This number, denoted $n!$ is read “n factorial”.

An r-permutation of a set S is an ordering of r distinct elements of S. If there are r elements in the set, then the number of r-permutations is $r!$ like above, but suppose $\mid S \mid = n$ and $r < n$. In this case, constructing an r permutation, there are n possible choices for the first element of the ordering, so there are n 1-permutations. There are $n \times (n-1)$ possible 2-permutations, $n \times (n-1) \times (n-2)$ possible 3-permutations, and so on. For each successive element of the sequence, the number of possibilities decreases by 1 which gives

$$n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!}$$

possible r permutations.

An Example

In this example, we will determine the number of different passwords possible for an on-line account. How many passwords are possible if

- characters can only be used once
- each password can contain a number 0-9, a lower case letter, or an upper case letter
- each password is six characters in length

What happens if we allow characters to be repeated?

Solution

There are 62 possible characters: 26 lower case letters + 26 upper case letters + 10 numbers = 62 total characters. From this group we must order six such that the above conditions are met. This is given by the quantity $P(62, 6)$, or

$$\begin{aligned} P(62,6) &= \frac{62!}{(62-6)!} \\ &= \frac{62!}{56!} \\ &= \frac{62 \cdot 61 \cdot 60 \cdot 59 \cdot 58 \cdot 57 \cdot 56!}{56!} \\ &= 4.42 \times 10^{10} \\ &\approx 44,261,653,680. \end{aligned}$$

So there are 44 billion passcodes possible. Note, that if the characters can be repeated, then using the multiplication rule, there are 62 choices for each of the 6 password slots. This gives

$$62 \cdot 62 \cdot 62 \cdot 62 \cdot 62 \cdot 62 = 62^6 = 5.68 \times 10^{10}.$$

Polling Question #5

The domain names three letters long, like `www.ted.com` or `www.mtv.com` are extremely valuable. How many different three letter domain names are possible?

A. 26^3

B. $P(26, 3)$

C. $\dfrac{26!}{23!}$

D. $26 \cdot 25 \cdot 24$

Polling Question #6

Major league baseball teams are allowed to carry 25 players on the active roster. Suppose a team has 12 pitchers and 13 position players. Position players may bat anywhere from the first through the eighth slots and a pitcher will bat ninth. Under these conditions, how many different batting orders could a manager create?

A. 5,589,762,048

B. 259,459,200

C. 622,702,080

D. 51,891,840

Permutations vs. Combinations

One of the hardest things about working with permutations is distinguishing them from a similar concept called combinations. This example will hopefully show how they are different.

(a) How many ways are there to rearrange the letters ABCD choosing 3 at a time?

(b) How many different ways are there to choose a 3 element subset of ABCD?

ABC	ABD	ACD	BCD
ACB	ADB	ADC	BDC
BAC	BAD	CAD	CBD
BCA	BDA	CDA	CDB
CAB	DAB	DAC	DBC
CBA	DBA	DCA	DCB
This column contains the ABC combination	This column contains the ABD combination	This column contains the ACD combination	This column contains the BCD combination

There are four combinations corresponding to the four columns. There are 24 total permutations corresponding to each reordering of three letters taken from ABCD. This can be calculated from the multiplication rule as 4 x 3 x 2 = 24.

Connection to Combinations

As long as $0 \leq r \leq n$, it can be shown that

$$C(n,r) = \binom{n}{r} = \frac{n!}{(n-r)!r!} = \frac{P(n,r)}{r!}.$$

The notation $\binom{n}{r}$ is called a binomial coefficient and is a shortcut notation for working with combinations.

How do we explain the connection between the combination and permutation formulas?

Imagine the problem of selecting a group of three runners to finish 1st through 3rd from a field of ten runners. Since order matters in this problem there are two steps involved. First you must choose your top 3. This can be done in $C(10,3)$ ways. Then you must order your top 3 and this can be done in 3! ways. Altogether this gives,

$$C(10,3) \cdot 3! = P(10,3).$$

This shows the relationship mentioned above.

Counting Cards

In five card study poker you are dealt five cards from a traditional 52 card deck. A traditional poker deck contains:

- 4 suites (clubs $\clubsuit$, spades $\spadesuit$, hearts $\heartsuit$, diamonds $\diamondsuit$)
- 13 cards per suite (called a face value) (2, 3, . . . , 10, J, Q, K, A)
- spades and clubs are black, diamonds and hearts are red

1. How many distinct five card hands are possible?
   Solution: Because there are 52 cards in a deck and we need five cards of any type, in any order to create a hand there are

$$C(52,5) = \binom{52}{5} = 2,598,960$$

   possible starting hands in five card poker.

2. How many different starting hands result in a four of kind?
   Solution: Pick the face value for the four of a kind (i.e. fives). There are 13 ways to do this. Next, choose the four cards from that face value there are $C(4,4) = 1$ one way to do this. Finally, pick the last card to fill out the hand. There are 48 cards left to choose from. Altogether there are $13 \times 1 \times 48 = 624$ four of a kind hands.

Permutations with Repeated Elements

Sometimes it is not enough to be able to count just permutations or combinations because each of these assumes the objects you are counting are distinguishable from one another. What do we mean by this? Let's consider the following example as an illustration.

How many distinct rearrangements of the word Mississippi are possible?

Constructing the ordering of the letters can be thought of as a four step process. (1) Choose a subset of four positions for the S's. (2) Choose a subset of four positions for the I's. (3) Choose a subset of two positions for the P's. (4) Choose a subset of one position for the M.

By the multiplication rule:

$$\text{number of rearrangements} = \binom{11}{4}\binom{7}{4}\binom{3}{2}\binom{1}{1} = \frac{11!}{4! \cdot 4! \cdot 2! \cdot 1!} = 34,650.$$

Class Discussion Question #1

Example

Janine's boss has allowed her to have a flexible schedule. Her boss says she can pick any five days of the week to work.

(a) How many different work schedules can be made?

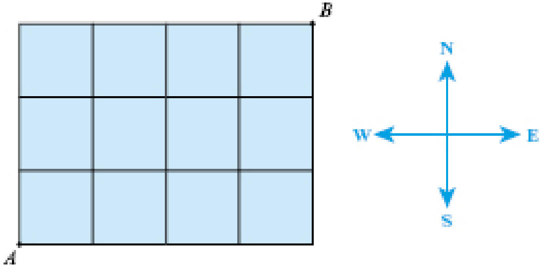
(b) How many choices give her consecutive days off?

(c) How many choices give her Wednesday off?

Class Discussion Question #2

Sometimes it is not enough to be able to count just permutations or combinations because each of these assumes the objects you are counting are distinguishable from one another. What do we mean by this? Let's consider the following example as an illustration.

Determine the number of pathways from A to B in the following grid if you can only move north or east.



Solution

To go from A to B, you must travel four blocks east (E) and 3 blocks north (N). Therefore, the number of paths from A to B will correspond to the number of distinguishable arrangements of E E E E N N N. Therefore, the number of paths is given by

$$\frac{7!}{4!3!} = 35.$$

Why is this correct? Well consider, the constructing of the ordering for the letters can be thought of as a two step process: (1) choose a subset of four positions for the E's, (2) choose a subset of 3 positions for the N's. Since there are 7 total positions in this chain, there are $\binom{7}{4}$ subsets for the E's. This leaves $\binom{3}{3}$ positions left to place the N's, but $\binom{3}{3} = 1$ so our choice is essentially made for us. But consider that

$$\binom{7}{4} \cdot \binom{3}{3} = \frac{7!}{4!3!} \cdot \frac{3!}{3!0!} = \frac{7!}{4!3!}$$

as promised.

Polling Question #7

Example

In a garden, seven flowers are to be arranged around a circular walk. Two arrangements of the flowers are considered different only when the positions of the flowers change relative to one another. What is the total number of different possible arrangements of the flowers?

Solution

If the flowers were to be arranged in a row then the answer is quite clearly $7 = 5040$, but in a circular arrangement this counts the following 7 arrangements as being different, when clearly they are not:

ABCDEFG
BCDEFGA
CDEFGAB
DEFGABC
EFGABCD
FGABCDE
GABCDEF

Because there are 7 such arrangements the real answer is $7!/7 = 6! = 720$, because what you are really ordering are the six positions after choosing the first which can't be chosen in any unique way.