Calculating Areas

Section 6.1

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Fall 2012

Measuring Area By Slicing

We first defined the integral in terms of the area bounded by a single function and the x-axis. Not surprisingly, integrals can also be used to measure areas of more general planar regions, bounded by two, three, and more graphs. Not surprisingly the basic idea comes back to approximations with rectangles.

Example

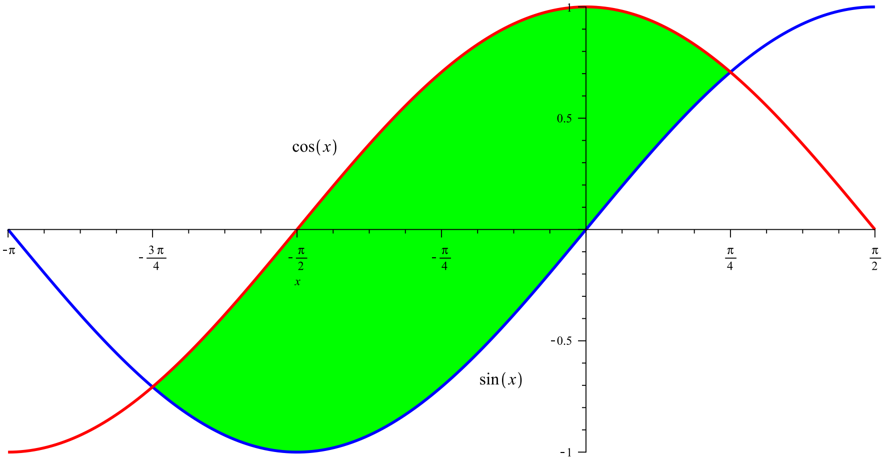
Approximate the area trapped between the curves $y = \sin x$ and $y = \cos x$ from $x = -3\pi/4$ and $x = \pi/4$.

Remarks:

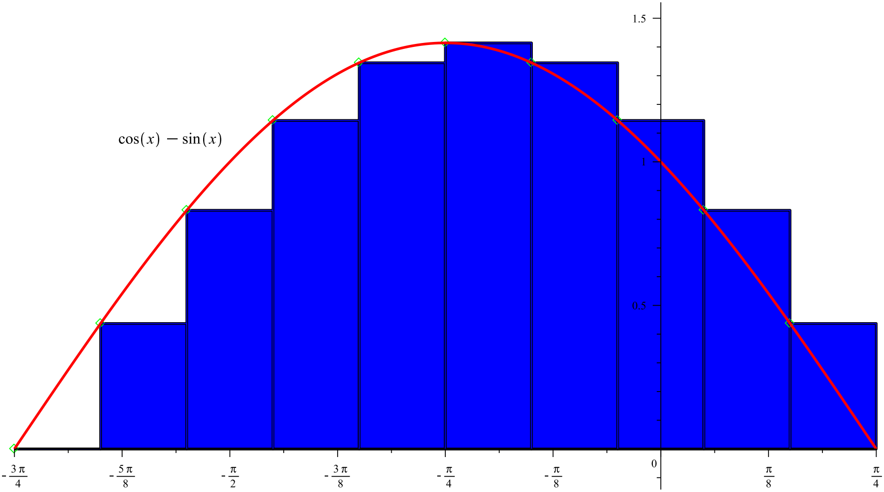
- The curves intersect at $x = -3\pi/4$ and $x = \pi/4$. The height of each rectangle is determined at its left edge, $x = x_i$; the height is $\cos(x_i) - \sin(x_i)$.
- The total area of ten rectangles is L_{10}, the left sum with ten equal subdivisions for the function $\cos x - \sin x$ on the interval $[-3\pi/4, \pi/4]$.
- The area between the curves is the integral

$$\int_{-3\pi/4}^{\pi/4} (\cos x - \sin x)\, dx = 2\sqrt{2}.$$

The Area in Question



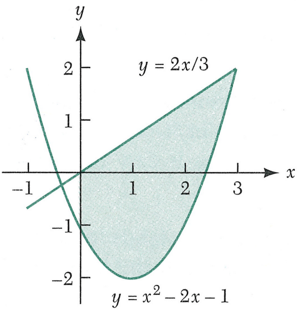
What We are Really Calculating



Finding Areas by Vertical Slices

Example

Find the area bounded by the graphs of $y = 2x/3$, $y = x^2 - 2x - 1$ and the y-axis.



Solution: We must start by finding the intersection between the graphs $y = 2/3x$ and $y = x^2 - 2x - 1$. This occurs at $x = 3$. We integrate the top and bottom curves and obtain

$$\text{Area} = \int_0^3 \left(\frac{2}{3}x - (x^2 - 2x - 1) \right) \, dx = 6.$$

A Slightly Harder Example

Example

Find the area of the region enclosed by $x = y^2$ and $y = x - 2$.

Solution: We start by finding where the curves intersect. This yields

$$y^2 = y + 2 \Longrightarrow y^2 - y - 2 = 0 \Longrightarrow (y+1)(y-2) = 0$$

from which we obtain $y = -1, y = 2$. Substituting into either equation we see that the corresponding x-values are $x = 1$ and $x = 4$, so the intersection points are $(1, -1)$ and $(4, 2)$.

Vertical Slices: We split the region into two parts A_1, A_2 and find the area of each part separately. We have $f(x) = \sqrt{x}, g(x) = -\sqrt{x}, a = 0, b = 1$ and so

$$A_1 = \int_0^1 \sqrt{x} - (-\sqrt{x})\ dx = 2\int_0^1 \sqrt{x}\ dx = \frac{4}{3}.$$

For A_2 we have $f(x) = \sqrt{x}, g(x) = x - 2, a = 1, b = 4$, so

$$A_2 = \int_1^4 \sqrt{x} - (x-2)\ dx = \frac{19}{6}.$$

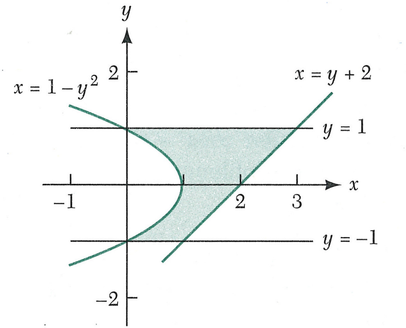
Horizontal Slices: If we integrate horizontally we do not have to split the region and $r(y) = y + 2$ and $l(y) = y^2$ so the area A for the region is given by

$$A = \int_{-1}^2 r(y) - l(y)\ dx = \int_{-1}^2 y + 2 - y^2\ dy = \frac{27}{6}.$$

Finding Areas by Horizontal Slices

Example

Find the area bounded by the graphs of $y = x - 2$, $x = 1 - y^2$, and the lines $y = \pm 1$.



Solution: Because this region does not have consistent rectangles if we slice vertically, it makes sense to write the bounding functions in terms of y and integrate across the y-axis. This means

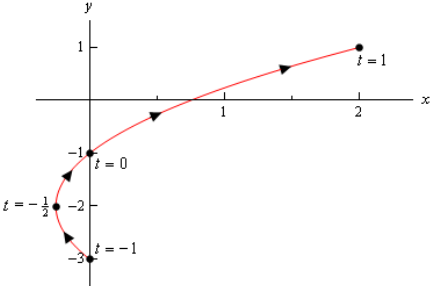
$$\text{Area} = \int_{-1}^{1} (y + 2 - (1 - y^2))\, dy = 8/3.$$

Parametric Curves

We have explored how to find areas by taking vertical slices and horizontal slices (rectangular thinking) and by taking circular wedges (polar thinking). In this lecture we will extend these techniques to deal with another variable change that leads to *parametric equations*. The basic idea behind parametric curves is that we write the x, y coordinates in terms of a parameter t (usually time). In symbols,

$$x = x(t)$$
$$y = y(t)$$

The basic idea behind the formation of a parametric curve is that of a directed trajectory.



Parameterizing a Line Segment

Example

Find a parametrization for the line segment with endpoints $(-2, 1)$ and $(3, 5)$.

Solution: Using $(-2, 1)$ as the starting point we create the parametric equations

$$x = -2 + at, \qquad y = 1 + bt.$$

We determine a and b so that the line will go through $(3, 5)$ when $t = 1$. Solving each equation for t and equating the results we obtain

$$\frac{x+2}{a} = \frac{y-1}{b}.$$

This leads to

$$\begin{aligned} 3 = -2 + a &\Longrightarrow a = 5 \\ 5 = 1 + b &\Longrightarrow b = 4 \end{aligned}$$

Therefore,

$$x = -2 + 5t, \qquad y = 1 + 4t, \qquad 0 \le t \le 1$$

is a parametrization of the line segment with initial point $(-2, 1)$ and terminal point $(3, 5)$.

General Form of the Parametrization of a Line

Parametrization of a Line Segment

A line with starting point (α, β) and terminal point (γ, δ) can be parameterized by the equations

$$x = \gamma t + \alpha(1 - t), \qquad \delta t + \beta(1 - t), \qquad 0 \leq t \leq 1.$$

Solution: If we look at the previous example with starting point $(\alpha, \beta) = (-2, 1)$ and terminal point $(\gamma, \delta) = (3, 5)$ we obtain the parametrization

$$x = 3t - 2(1 - t), \qquad y = 5t + 1(1 - t), \qquad 0 \leq t \leq 1.$$

After collecting terms these equations agree with our previous answer.

Standard Parameterizations

Circle

A parametrization of the circle $x^2 + y^2 = a^2$ is given by

$$x = a\cos t, \qquad y = a\sin t, \qquad 0 \le t \le 2\pi.$$

Ellipse

A parametrization of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is given by

$$x = a\cos t, \qquad b\sin t, \qquad 0 \le t \le 2\pi$$

Standard Parametrization

Every function $y = f(x)$ admits a standard parametrization given by

$$x = t, \qquad y = f(t)$$

where the domain of t is determined by the domain of $f(x)$. Similarly, the parametrization of the inverse of $f(x)$ is given by

$$x = f(t), \qquad y = t$$

where the domain of t is determined by the range of $f(x)$.

Shapes Created Via Rolling Wheels

Many fascinating curves are generated by points on rolling wheels. The path of light on the rim of a rolling wheel is a *cycloid*, which has parametric equations

$$x = a(t - \sin t), \quad y = a(1 - \cos t), \qquad t \geq 0$$

where $a > 0$.

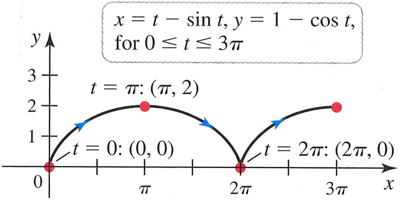
Another equally interesting curve is the one generated by a circle with radius $a/4$ that rolls along the inside of a larger circle with radius a. The curve created in this case is called an *astroid* or *hypocycloid*, and its parametric equations are

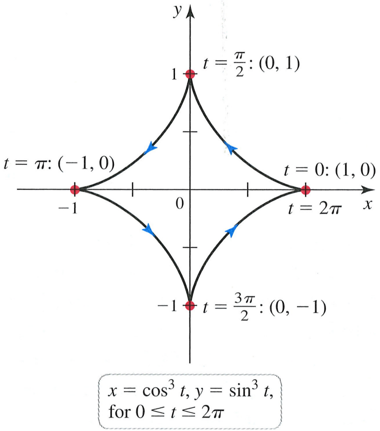
$$x = a\cos^3 t, \quad y = a\sin^3 t, \qquad 0 \leq t \leq 2\pi.$$

Example

Find the rectangular forms of each of these parametric equations and sketch their graphs. Label the graphs with the parameter t.

Graphs of the Cycloid and Asteroid





Rectangular Equations

Cycloid: $x = a\cos^{-1}\left(1 - \frac{y}{a}\right) - \sqrt{2ay - y^2}$

Asteroid: $x^{2/3} + y^{2/3} = 1$

Calculating Areas Enclosed by Parametric Curves

As we are now no doubt aware of the area under a curve $y = h(x)$ from a to b is given by the integral

$$\text{Area} = \int_a^b h(x)\,dx.$$

We think of $h(x)$ as a height function, but if this function is defined by a parametric equation $x = f(t)$ and $y = g(t)$, then a simple substitution allows us to calculate area in the following way:

$$\text{Area} = \int_a^b y\,dx = \int g(t)f'(t)\,dt$$

where the limits of integration must be determined by inspection.

Area Enclosed by the Hypocycloid

Example

Find the area enclosed by the hypocycloid, $x^{2/3} + y^{2/3} = 1$. Recall the parametrization of this curve is given by

$$x = a\cos^3 t, \quad y = a\sin^3 t, \qquad 0 \le t \le 2\pi.$$

Solution: Applying the formula from the previous slide we calculate the area enclosed in one quadrant and use symmetry to find the full answer. The area enclosed in quadrant I is given by

$$\text{Area} = \int_0^a y\,dx = \int_0^{\pi/2} (a\sin^3 t)(3a\cos^2 t \cdot -\sin t)\,dt = \frac{3}{32}\pi a^2.$$

Multiplying the result by 4 for the full area gives

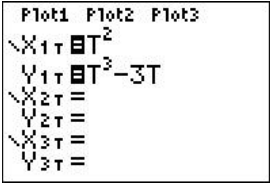
$$\text{Area of a Hypocycloid} = \frac{3}{8}\pi a^2.$$

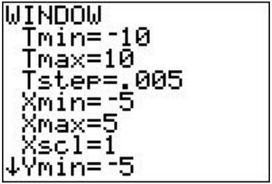
A Slightly Harder Example

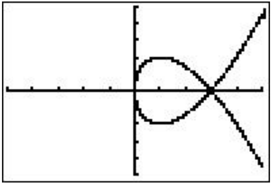
Example

Find the area bounded by the loop of the curve with parametric equation $x = t^2$, $y = t^3 - 3t$.

Solution: The hardest part of finding areas involving parametric equations is determining the range over which t runs. We usually must resort to graphing to help as shown in the screen shots below.







Solution

From the equation it is clear that when $t = 0$, $(x, y) = (0, 0)$. This implies the graph begins at the origin, but to form the loop we need to know the values of t that result in the intersection. This would clearly be a value of t where $y = 0$ based on the graph, so we solve

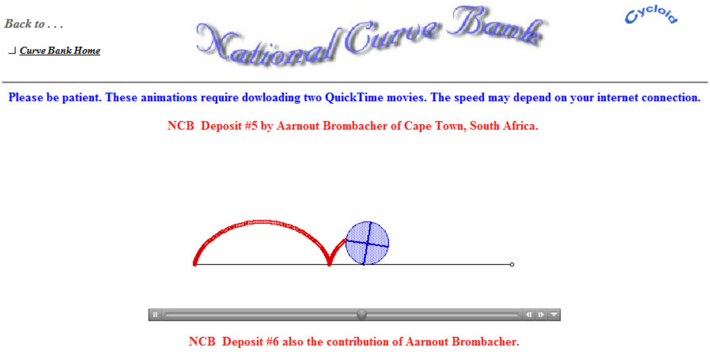
$$t^3 - 3t = 0 \Longrightarrow t(t^2 - 3) = 0 \Longrightarrow t = \pm\sqrt{3}.$$

We can recover the area enclosed by finding the top (or bottom half area) and using symmetry. The area enclosed by the curve is given by

$$\text{Area} = 2\int_0^3 y\,dx = 2\int_0^{-\sqrt{3}} (t^3 - 3t)(2t)\,dt = \frac{24}{5}\sqrt{3} \approx 8.31.$$

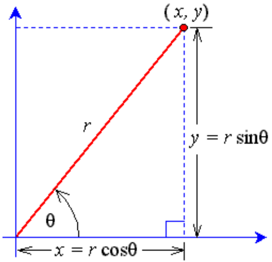
National Curve Bank

The National Curve Bank is a website supported by the National Science Foundation (NSF) and the Beckman foundation and is run by California State University (LA). The website is located at `http://curvebank.calstatela.edu/home/home.htm` and is home to a wide array of animations and graphs of curves arising in all sorts of mathematical applications.



Polar Coordinates

One of the most important coordinate systems we work with in mathematics is the polar coordinate system. Polar coordinates are inspired by our work with trigonometric functions and work best when trying to analyze annular regions (or regions involving circles or ellipses).



The basic idea is that given our familiarity with $\sin\theta$ and $\cos\theta$ we are able to completely redefine the way we think of plotting points in the plane. Our regular coordinate system is called a *rectangular coordinate system*. In this lecture, we will introduce the *polar coordinate system* and explore its application to finding areas of annular regions.

$$r = \text{distance to the origin} \qquad x = r\cos\theta \qquad r = \pm\sqrt{x^2 - y^2}$$

$$\theta = \text{standard angle rotation} \qquad y = r\sin\theta \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Plotting Points in Polar Coordinates

Consider plotting the point $(x, y) = (1, -1)$ in polar coordinates. What are the coordinates which describe this exactly?

The problem with the term exactly implies there is only one answer, and when it comes to polar coordinates that is simply just not the case. In this small example, the point $(1, -1)$ in rectangular coordinates can be described by

$$(r, \theta) = (-\sqrt{2}, 3\pi/4), (\sqrt{2}, 7\pi/4), (\sqrt{2}, -\pi/4)$$

just to name a few.

$\boxed{\text{So how do we handle this ambiguity?}}$

The typical convention for polar coordinates is that the variable r and θ obey the following:

1. $0 \leq r < \infty$
2. $-\pi \leq \theta \leq \pi$
3. $0 \leq \theta \leq 2\pi$

As you can see there is not even a consensus on the range for θ, but we'll generally use $0 \leq \theta \leq 2\pi$ unless otherwise indicated.

Coordinate Comparisons

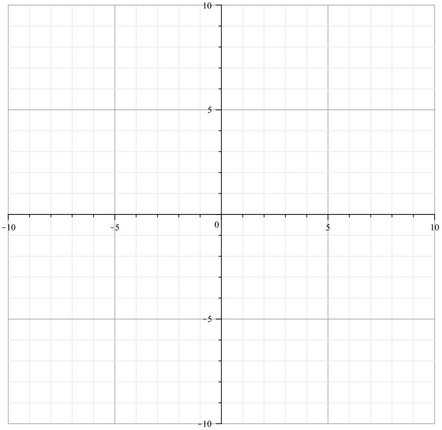


Figure: Rectangular Coordinate System

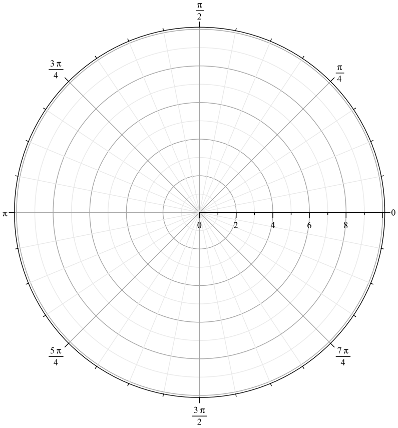
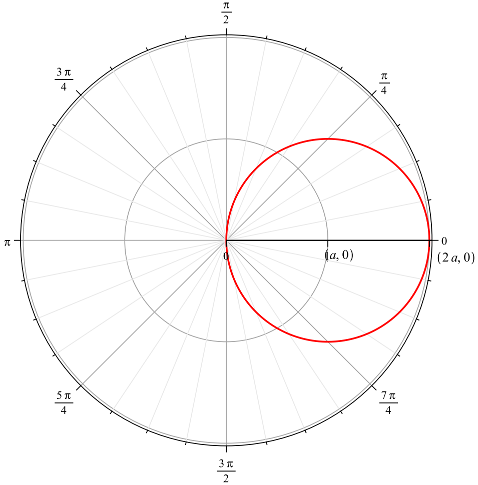


Figure: Polar Coordinate System

From Polar to Rectangular and Back Again

Consider the function in the polar coordinate system determined by the graph below. What is the equation in polar coordinates? In rectangular coordinates?



Solution continued...

Looking at the graph on the previous slide we should notice this as a circle of radius a centered at $(a, 0)$. In rectangular coordinates, this gives

$$(x-a)^2 + y^2 = a^2.$$

We now look to convert this expression to polar coordinates. Notice that if we expand the left hand side we obtain

$$x^2 - 2ax + a^2 + y^2 = a^2.$$

This means

$$x^2 + y^2 - 2ax = 0 \Longrightarrow r^2 - 2a(r\cos\theta) = 0.$$

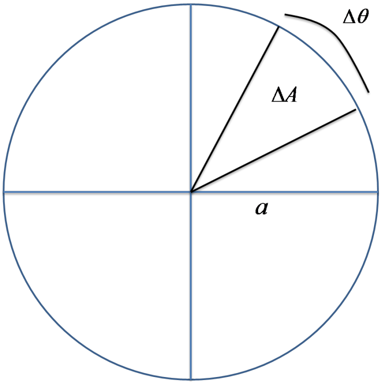
Finally, we obtain the expression in polar coordinates, $r = r(\theta)$, given by

$$r(\theta) = 2a\cos\theta.$$

You should note that when $\theta = 0$ we are at the point $(2a, 0)$ and when $\theta = \pm\pi/2$ we are at the point $(0, 0)$. This implies the angle θ takes on the range $\theta \in [-\pi/2, \pi/2]$ if we want the angle drawn out to use positive radii the whole way through which was one of our conventions stated earlier.

Calculating Area Using Polar Coordinates

In order to get a feel for how to calculate areas using polar coordinates we consider the basic problem of finding the area of a circle of radius a. The basic approach involves summing wedges as opposed to rectangles like we did with Riemann sums.



It is pretty easy to find the area of this pie wedge as it is simply a fraction of the total area of the circle which we know to be $A = \pi a^2$. Looking at ΔA, we have

$$\Delta A = \frac{\Delta\theta}{2\pi} \cdot \pi a^2.$$

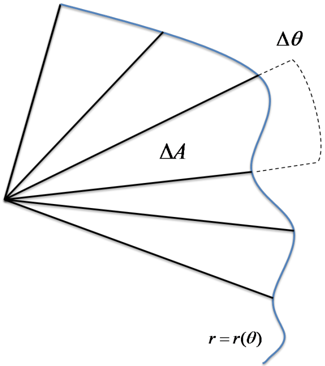
This means that the formula for the area of these wedges is given by

$$\Delta A = \frac{1}{2} a^2 \Delta\theta.$$

This is also the formula for the area of a sector of a circle.

The Variable Pie

As we know, not all regions in the plane are nice circles like this, but we've established a basic manner of thought here, that can actually pay dividends when applied to a more general situation. Consider the quadrant of a pie with variable boundary given by $r = r(\theta)$. How do we find the area of this region?



From our previous experience with the circle, we know that in terms of $r(\theta)$, the area ΔA of the adjusted wedge is

$$\Delta A = \frac{1}{2} r^2 \Delta\theta.$$

Summing over all such wedges and letting $\Delta\theta \to 0$, we obtain the integral formula for the total area over the annular region $\theta_1 \leq \theta \leq \theta_2$,

$$\int_{\theta_1}^{\theta_2} \frac{1}{2} r(\theta)^2 \, d\theta.$$

An Example

Example

Find the area of the circle centered at $(a, 0)$ of radius a given by $r(\theta) = 2a\cos\theta$.

Solution: It is important to note that this is the same figure considered earlier in this lecture. In this case, the variable radius is given by $r(\theta) = 2a\cos\theta$, so applying our polar area formula, we have

$$\begin{aligned}
\text{Area} &= \int_{-\pi/2}^{\pi/2} \frac{1}{2}(2a\cos\theta)^2 \, d\theta \\
&= 2a^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta \, d\theta \\
&= 2a^2 \int_{-\pi/2}^{\pi/2} \frac{1+\cos(2\theta)}{2} \, d\theta \\
&= a^2 \left(\theta + \frac{1}{2}\sin(2\theta)\right)\Bigg|_{-\pi/2}^{\pi/2} \\
&= \pi a^2
\end{aligned}$$

Which is exactly the area we expected from basic geometry.

An Example With Symmetries

Example

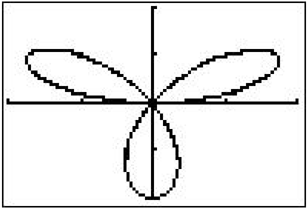
Find the entire area enclosed by the curve $r = 2\sin 3\theta$.

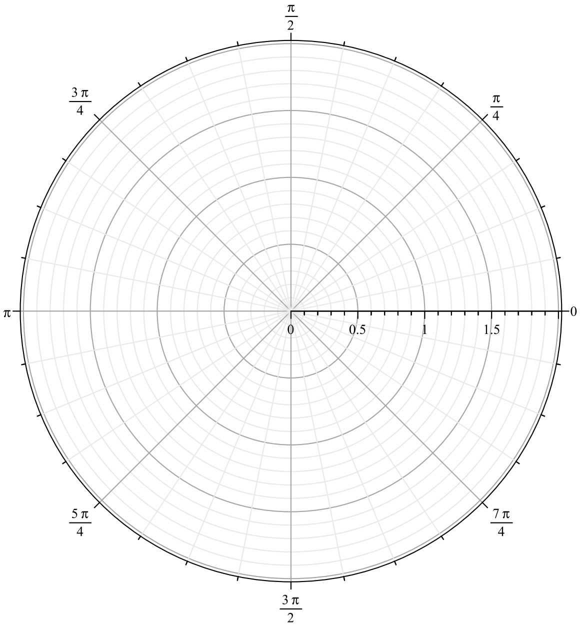
Solution: We will begin by drawing the curve (using the polar graph on the next slide and plotting points). In fact, in all area problems in polar coordinates a sketch should be made. The curve is called a *rose curve*. A single petal is described completely as θ goes from $0 \to \pi/3$. This means we can use symmetries and find the area of this loop only. We have

$$\frac{A}{3} = \frac{1}{2}\int_0^{\pi/3} (2\sin 3\theta)^2 \, d\theta \Longrightarrow A = 6\int_0^{\pi/3} \frac{1-\cos 6\theta}{2} \, d\theta.$$

Integration yields,

$$A = 6\left[\frac{\theta}{2} - \frac{1}{12}\sin 6\theta\right]_0^{\pi/3} = \pi.$$





π/2
π/4
0
3π/4
π
5π/4
3π/2
7π/4
0
0.5
1
1.5

Intersecting Polar Curves

Example

Find the area inside the circle $r = 5\cos\theta$ and outside the curve $r = 2 + \cos\theta$.

Solution: The two curves intersect when

$$5\cos\theta = 2 + \cos\theta \iff \cos\theta = 1/2.$$

This means $\theta = \pm\pi/3$. The area inside $5\cos\theta$ and outside $2 + \cos\theta$ is given by

$$A = \frac{1}{2}\int_{-\pi/3}^{\pi/3} (5\cos\theta)^2 \, d\theta - \frac{1}{2}\int_{-\pi/3}^{\pi/3} (2 + \cos\theta)^2 \, d\theta.$$

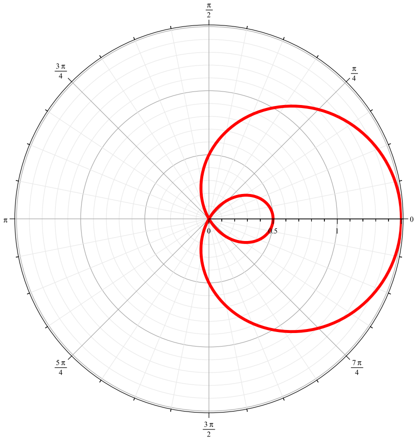
We observe that both curves are symmetric to the x-axis and the integrals may be combined since the limits of integration are the same. This gives,

$$\begin{aligned} A &= \int_0^{\pi/3} [25\cos^2\theta - (4 + 4\cos\theta + \cos^2\theta)] \, d\theta \\ &= [8\theta + 6\sin 2\theta - 4\sin\theta]_0^{\pi/3} = \frac{8\pi}{3} + \sqrt{3}. \end{aligned}$$

A Harder Example

Example

Find the area that lies inside the larger loop and outside the smaller loop of the limacon, $r = \frac{1}{2} + \cos\theta$.



Solution...

The trick to finding the solution in this case is figuring out the value of θ for which certain parts are drawn and what the radius looks like for those parts. We use the symmetry across the x-axis by finding only the area trapped in the top half and then multiplying by 2. For $0 \leq \theta \leq 2\pi/3$, the top lobe is drawn out. We need to then subtract the area contained in the smaller loop which is drawn out for $\pi \leq \theta \leq 4\pi/3$. All together (remembering to multiply by 2) we have

$$\begin{aligned} A &= 2\left[\int_0^{2\pi/3} \frac{1}{2} r^2 d\theta - \int_\pi^{4\pi/3} \frac{1}{2} r^2 d\theta\right] \\ &= \int_0^{2\pi/3} \left(\frac{1}{2} + \cos\theta\right)^2 \, d\theta - \int_\pi^{4\pi/3} \left(\frac{1}{2} + \cos\theta\right)^2 \, d\theta \\ &\approx 2.08443 \end{aligned}$$