Expected Value

Section 14.4-14.5

Dr. John Ehrke
Department of Mathematics

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Expected Value

Expected value is a mathematical way to use probabilities to determine what to expect in various situations over the long run. Expected value is used to

- determine premiums on insurance policies
- weigh risks versus benefits in alternative business ventures
- indicate to a player of any game of chance what will happen if the game is played repeatedly
- indicate the “average” or “expected” result of any stochastic result

Outcome Table

The standard way to compute expected value involves the use of an outcome table.

Example

Consider all three child families. Assuming all children are only boys or girls, and there is an equally likely chance of a family having a boy as a girl. What is the expected value of the number of girls in a three child family?

Outcome	Numeric Assignment, x_i	Probability, $P(x_i)$	$x_i \cdot P(x_i)$
no girls	0	$\frac{1}{8}$	$0 \cdot \frac{1}{8} = 0$
one girl	1	$\frac{3}{8}$	$1 \cdot \frac{3}{8} = \frac{3}{8}$
two girls	2	$\frac{3}{8}$	$2 \cdot \frac{3}{8} = \frac{6}{8}$
three girls	3	$\frac{1}{8}$	$3 \cdot \frac{1}{8} = \frac{3}{8}$

Remarks

- The expected value of the number of girls in a three child family is 1.5.
- This answer makes sense in the respect that under these conditions it says on average half the children will be boys and half will be girls.
- The value 1.5 makes no sense in reality though as 1.5 children of any gender would be somewhat problematic.
- This means that expected values are not necessarily outcomes of the stochastic processes they model. Their values must be interpreted in the context of the problem.

Fairness

Example

A game is played by selecting one bill at random from a bag containing 10 \$1 bills, 5 \$2 bills, three \$5 bills, one \$10 bill, and one \$100 bill. The player gets to keep the selected bill. There is a \$10 charge to play. What is the expected value of the game? Is the game fair? What should the charge for the game be in order for the game to be fair?

The idea of fairness as it relates to expected value means neither side in the long run profits from playing the game repeatedly. This would suggest that the expected value of a fair game is 0.

Payoff Table

The payoff table for the previous problem is included below:

Outcome	Numeric Assignment, x_i	Probability, $P(x_i)$	$x_i \cdot P(x_i)$
choose a \$1	\$1 - \$10 = -\$9	10/20	$-9 \cdot 0.5 = -4.5$
choose a \$2	\$2 - \$10 = -\$8	5/20	$-8 \cdot 0.25 = -2$
choose a \$5	\$5 - \$10 = -\$5	3/20	$-5 \cdot 0.15 = -0.75$
choose a \$10	\$10 - \$10 = \$0	1/20	$0 \cdot 0.05 = 0$
choose a \$100	\$100 - \$10 = \$90	1/20	$90 \cdot 0.05 = 4.5$

So the expected payoff is $-4.5 + -2 + -0.75 + 0 + 4.5 = -2.75$. This means on average, everytime you play you are throwing away \$2.75. This game is not fair and is weighted in favor of the house.

Determining a Split

Example

Al and Betsy played a coin tossing game in which a fair coin was tossed until a total of either three heads or three tails occurred. Al was to win if three heads were flipped and Betsy would win if three tails were flipped. Each bet $50. If the coin was lost after flipping two heads and one tail, how should the stakes be fairly split if the game is not continued?

Solution: In this case, since Al is ahead, he will win 3/4 of the remaining time (confirm in a tree diagram) and Betsy will win 1/4 of the time. This suggests the only fair way to split the remaining money is to award Al $75 and Betsy $25. These values agree with their expected values. The results are summarized below from Al's perspective:

Outcome	Numeric Assignment, x_i	Probability, $P(x_i)$	$x_i \cdot P(x_i)$
Al wins	$100	3/4	$0.75 \cdot 100 = \$75$
Al loses	$0	1/4	$0.25 \cdot 0 = \$0$

Insurance Premiums

> *The other day I was confronted with a dose of reality. My life expectancy is 74 years. My wife has a 79-year expectancy. Let's not get into whether men have shorter live (on average) because they are married to women. These are average, expectations. I doubt my life will abruptly terminate on my 74th birthday. The insurance world uses the averages for accident rates at a particular age, cost to repair a car, likelihood a certain model of car will be in a wreck, etc.. to calculate an expected value for each of us. Then they tack on the profit they need to keep the stockholders happy. And by the way, married men and women are expected to live longer than their single counterparts! That tidbit factors into the cost of your life insurance.*

Why does this matter? Let's look at an example:

Example

Back when I was just starting to drive, someone (read highly intelligent mathematician) assessed the probability of me smashing my car to be about .1025 per month. The average cost to repair my car was about $2300. Let's assume the insurance company wants to make a profit of 7%, how much should they charge for my monthly insurance premium?

Solution

We start by calculating the expected value of a claim set equal to $0, so that neither side is ahead. We will then tack 7% onto this value to satisfy the policy's profit requirements.

$$0.1025(-2300) + 0.8975(X) = 0$$

$$X = \frac{0.1025(-2300)}{0.8975} \approx 262.68$$

$$\text{Premium } = 1.07(262.68) = \$281.06$$

Polling Question #14

From experience, a shipping company knows that the cost of delivering a small package within 24 hours is $14.80. The company charges $15.50 for shipment, but guarantees to refund the charge if delivery is not made within 24 hours. If the company fails to deliver only 2% of its packages within the 24 hour window, what is the company's expected profit per package?

(a) $14.89

(b) $0.39

(c) $0.67

(d) $-14.49

Solution

Example

From experience, a shipping company knows that the cost of delivering a small package within 24 hours is \$14.80. The company charges \$15.50 for shipment, but guarantees to refund the charge if delivery is not made within 24 hours. If the company fails to deliver only 2% of its packages within the 24 hour window, what is the company's expected profit per package?

Outcome	Numeric Assignment, x_i	Probability, $P(x_i)$	$x_i \cdot P(x_i)$
On-Time Delivery	\$0.70	0.98	$0.70 \cdot 0.98 = 0.686$
Late Delivery	-\$14.80	0.02	$-14.80 \cdot 0.02 = -0.296$
		Expected Profit	$= 0.39$

The shipping company can expect to make \$0.39 per package on average.

Setting a Fair Price

Example

Suppose you roll two fair six sided dice. If you are playing a game where you are paid the sum of the dice showing, what price should the game be set at so that it is fair for both sides?

Solution: Without knowing what price the game is set at we have two options. We can either let x represent the game price or we can assume the price to play is \$0 and adjust our price based on the expected value of the game. We will choose the latter.

Solution

Outcome	Numeric Assignment, x_i	Probability, $P(x_i)$	$x_i \cdot P(x_i)$
2	2	1/36	2/36
3	3	2/36	6/36
4	4	3/36	12/36
5	5	4/36	20/36
6	6	5/36	30/36
7	7	6/36	42/36
8	8	5/36	40/36
9	9	4/36	36/36
10	10	3/36	30/36
11	11	2/36	22/36
12	12	1/36	12/36

Expected Value $= 7$

If the game was to be played with a \$0 price to play the player would expect to win \$7 on average. (This is not surprising, why?) Because of this result we would need to charge \$7 to play the game to make it fair.