Improper Integration, Limits at Infinity

Section 5.10

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Comparing Growth Rates

In this lecture we are tasked with investigating the behavior of integrals as part of a limiting process. To really understand this material we need to explore the idea of how fast or slow certain groups of functions grow relative to one another. We will demonstrate this idea with the following examples using L'Hopital's rule.

Example

Evaluate $\lim_{x\to\infty} \frac{e^{px}}{x^n}$ where $n, p > 0$.

Solution: We note that this of type ∞/∞. It seems strange to consider this problem at first, but we quickly note that pulling the power of n out can make this much easier.

$$\lim_{x\to\infty} \frac{e^{px}}{x^n} = \left(\lim_{x\to\infty} \frac{e^{px/n}}{x}\right)^n.$$

Applying L'Hopital's rule at this point gives

$$\left(\lim_{x\to\infty} \frac{e^{px/n}}{x}\right)^n = \lim_{x\to\infty} \left(\frac{\frac{p}{n}e^{px/n}}{1}\right)^n = \lim_{x\to\infty} \left(\frac{pe^{px}}{n^n}\right) = \infty.$$

This example implies that the exponential function grows faster than any positive power of x as $x \to \infty$.

Another Growth Rate Example

Example

Evaluate $\lim_{x\to\infty} \frac{\ln x}{x^{1/p}}$ for $p > 0$.

Solution: This is of type ∞/∞. In this case we apply L'Hopital's rule straightforwardly and obtain

$$\lim_{x\to\infty} \frac{\ln x}{x^{1/p}} = \lim_{x\to\infty} \frac{1/x}{(1/p)x^{1/p-1}}.$$

Simplifying the expression on the right we have

$$\lim_{x\to\infty} \frac{1/x}{(1/p)x^{1/p-1}} = \lim_{x\to\infty} p \cdot x^{-1/p} \to 0.$$

This result holds for any $p > 0$ and so in particular for very large values of p or very small powers of x. This says that the $\ln x$ grows very slowly, in fact, more slowly than any positive power of x.

Rates of Growth and Decay, A Summary

In the literature you will often see a very specific notation when dealing with how functions grow (or decay) relative to other functions. In particular, when we write $f(x) << g(x)$, we mean that $f(x)$ grows much more slowly than $g(x)$, and in symbols is written:

$$f(x) << g(x) \Longrightarrow \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0.$$

Based on our observations in this lecture we can summarize the relative growth and decay rates of functions. The growth rates are given by

$$\underset{\text{slow}}{\ln x} << \underset{\text{moderate}}{x^p} << \underset{\text{fast}}{e^x} << \underset{\text{very fast}}{e^{x^2}}$$

where $p > 0$ and the decay rates are given by

$$\underset{\text{really slow}}{1/\ln x} >> \underset{\text{slow}}{1/x^p} >> \underset{\text{fast}}{e^{-x}} >> \underset{\text{very fast}}{e^{-x^2}}$$

Improper Integrals of Type 1

In this lecture we will deal with definite integrals which have one or both limits of integration set to $\pm\infty$. Integrals of this type are called *improper integrals*, and are defined in the following way.

Improper Integrals

If $\int_a^\infty f(x)\,dx$ exists for all $x \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{N\to\infty} \int_a^N f(x)\,dx$$

provided this limit exists. Similarly, if $\int_{-\infty}^b f(x)\,dx$ exists for all $x \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{N\to-\infty} \int_N^b f(x)\,dx$$

provided this limit exists. The improper integrals above are called convergent if the corresponding limit exists and divergent if the limit does not exist. Finite sums and differences of convergent integrals converge.

An Example

Example

Evaluate the improper integral $\int_0^\infty e^{-px}\, dx$ for $p > 0$.

Solution: We proceed to integrate and then we will take the limit,

$$\int_0^N e^{-px}\, dx = -\frac{1}{p}e^{-px}\Big|_0^N = -\frac{1}{p}e^{-pN} - \left(-\frac{1}{p}\right).$$

As $N \to \infty$, we obtain

$$\lim_{N\to\infty} -\frac{1}{p}e^{-pN} - \left(-\frac{1}{p}\right) = 1/p.$$

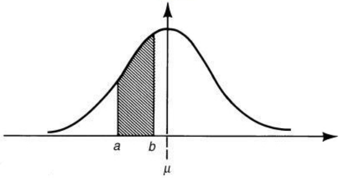
So this integral converges for every positive value of p.

Chasing Fat Tails

Despite the somewhat humorous title of this slide, we are indeed concerned with understanding whether the area under a function eventually becomes infinite or stays finite. Much of this concern can be tracked to the following (very important result) which we will not prove in this class.

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}.$$

This integral defines the well known normal density function pictured below.



Associated with every random variable X is a probability density function (like the one pictured above, such that

$$\int_a^b f(x)\, dx = \text{probability that } X \text{ lies in } (a, b).$$

Cases for the Tails

We will look at several examples of the same basic integral,

$$\int_1^\infty \frac{dx}{x^p}$$

for $p > 0$. These integrals are called p-integrals.

Example

Evaluate the integral $\int_1^\infty \frac{dx}{x}$. This is the $p = 1$ case.

Solution: We take

$$\lim_{N\to\infty} \int_1^N \frac{dx}{x} = \ln x\Big|_1^N = \ln N - \ln 1 \to \infty$$

as $N \to \infty$. This means the integral diverges.

The $p = 2$ Case

Example

Evaluate the integral $\int_1^\infty \frac{dx}{x^2}$. This is the $p = 2$ case.

Solution: We calculate the integral directly and get

$$\int_1^N \frac{dx}{x^2} = -\frac{1}{x}\bigg|_1^N = 1 - \frac{1}{N}.$$

As $N \to \infty$, this quantity goes to 1. We say the integral converges to 1 in this case.

The General Case

Based on the previous two examples there must be some value of p between 1 and 2 that is the tipping point for convergence and divergence. We will look at the general case to decide.

Example

Evaluate the integral $\displaystyle\int_1^\infty \frac{dx}{x^p}$.

Solution: In this case, we calculate

$$\int_1^\infty \frac{dx}{x^p} = \lim_{N\to\infty} \frac{x^{-p+1}}{-p+1}\bigg|_1^N = \frac{x^{-p+1}}{-p+1} - \frac{1}{-p+1}.$$

We evaluate this limit in cases.

- When $p < 1$, the integral diverges with answer ∞.
- When $p > 1$, the integral converges with answer $\frac{1}{p-1}$.
- When $p = 1$, we have already shown the integral diverges.

Applications of p-Integrals

There are a number of ways in which we will use this result. The main application revolves around the idea of limit comparisons described in the theorem below.

Limit Comparison Theorem for Integrals

Let f and g be continuous functions. Suppose that for all $x \geq a$, $0 \leq f(x) \leq g(x)$, then if $\int_a^\infty g(x)\,dx$ converges, then so does $\int_a^\infty f(x)\,dx$, and

$$\int_a^\infty f(x)\,dx \leq \int_a^\infty g(x)\,dx.$$

If the smaller function's integral diverges, then so does the larger.

In pictures, this theorem makes the claim that if the area under the g-graph is finite, then so is the area under the lower f-graph. And conversely, if the area under the f-graph is infinite, then so is the area under the g-graph.

A Limit Comparison Example

Suppose we were asked to determine whether the integral

$$\int_0^\infty \frac{dx}{\sqrt{1+x^2}}$$

converges or diverges. We recognize this integral as one of those suitable for a trig substitution, but going to such lengths is unnecessary if we only care about the question of convergence. Based on the previous slide we draw the comparison that for $x \geq 1$,

$$\sqrt{1+x^2} < \sqrt{x^2+x^2} = \sqrt{2}x.$$

This means, that

$$\frac{1}{\sqrt{1+x^2}} > \frac{1}{\sqrt{2}x}.$$

So we can say that

$$\int_1^\infty \frac{dx}{\sqrt{1+x^2}} > \frac{1}{\sqrt{2}} \int_1^\infty \frac{dx}{x}.$$

The smallest of these integrals diverges though (the $p = 1$ case) and so all the larger ones do as well, including our original integral.

An Easier Way to Think About It

What we are really doing in the limit comparison tests is making a comparison about the long term behavior or asymptotic behavior of two functions. There is a very specific notation given in this situation. We say that two functions are *asymptotically equivalent*, denoted $f(x) \sim g(x)$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1.$$

This means in some sense that eventually the functions are the same. For the previous example, we can make the case that $\sqrt{1+x^2} \sim \sqrt{x^2} = x$ because

$$\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{\sqrt{x^2}} = \lim_{x\to\infty} \sqrt{\frac{1+x^2}{x^2}} = 1.$$

This means that

$$\frac{1}{\sqrt{1+x^2}} \sim \frac{1}{x}$$

and so because one diverges the other must.

Another Type of Impropriety

Another type of improper integral arises when the integrand has an infinite discontinuity at a limit of integration or at some point between the limits of integration. We will start by considering what can go wrong if we ignore this discontinuity.

Example

Evaluate $\displaystyle\int_{-1}^{1} \frac{dx}{x^2}$.

Solution: If we evaluate the integral directly we obtain

$$\int_{-1}^{1} \frac{dx}{x^2} = -\frac{1}{x}\bigg|_{-1}^{1} = -2.$$

Clearly this answer is ridiculous since the function $1/x^2$ is everywhere positive. We must have done something wrong. A simple glance at the graph suggests we might have overlooked the discontinuity at $x = 0$.

Improper Integrals of Type 2

Improper Integrals with Infinite Discontinuities

1. If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{N\to b^-} \int_a^N f(x)\,dx$$

provided this limit exists.

2. If f is continuous on $(a, b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{N\to a^+} \int_N^b f(x)\,dx$$

provided this limit exists. The integrals above are convergent if the limit exists and are otherwise divergent.

3. If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then $\int_a^b f(x)\,dx$ is said to be convergent.

How Things Can Go Wrong

Example

Evaluate the integral $\int_{-1}^{1} \frac{dx}{x^2}$, correctly this time!

Solution: Applying the previous theorem, we split the integral into two pieces according to

$$\int_{-1}^{0} \frac{dx}{x^2} \quad \text{and} \quad \int_{0}^{1} \frac{dx}{x^2}.$$

We will consider each piece separately. For the first piece, we have

$$\int_{-1}^{0} \frac{dx}{x^2} = \lim_{c \to 0^-} \int_{-1}^{c} \frac{dx}{x^2} = \lim_{c \to 0^-} -\frac{1}{c} - 1 = \infty.$$

So, since the first piece diverges, there is no need to consider the second piece.

Establishing a General Case

We want to consider, like we did in the previous lecture, a general rule of convergence for the integral

$$\int_0^1 \frac{dx}{x^p}$$

where $p > 0$. Working out this integral, we have

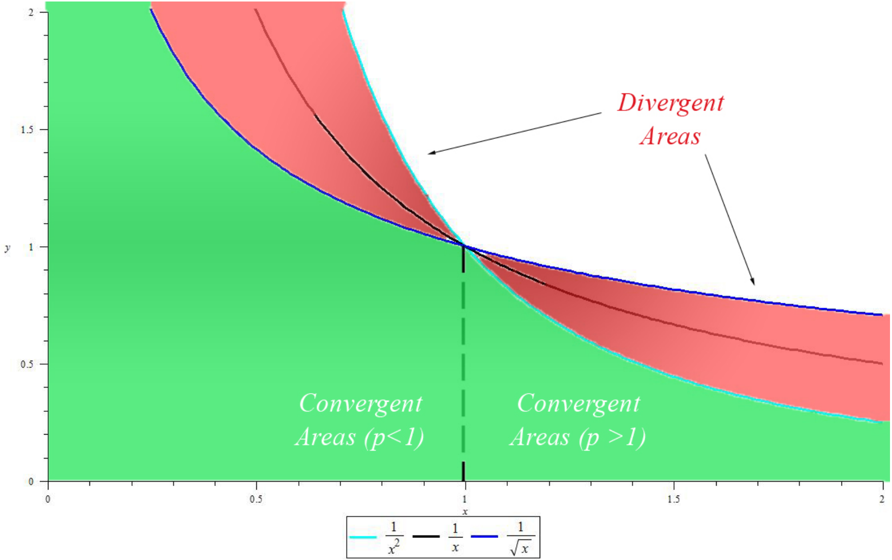
$$\int_0^1 \frac{dx}{x^p} = \lim_{N \to 0^+} \int_N^1 \frac{dx}{x^p} = \left. \frac{x^{1-p}}{1-p} \right|_N^1 = \frac{1}{1-p} - \frac{N^{1-p}}{1-p}.$$

Looking at this last expression when $N \to 0^+$, we have two cases:

- The integral converges when $p < 1$ and the answer is $\frac{1}{1-p}$.
- The integral diverges when $p \geq 1$.

Notice that this is just the opposite of the case when $x \to \infty$ with the $p = 1$ case being the crossover. There is a very nice picture of these relationships, which we consider in the next slide.

Comparisons of p-Integrals



Convergence of Improper Integrals Type 2

Using the graphical aid of the picture before, we want to close out this lecture by making sure you can quickly, and correctly identify whether a type 2 integral is convergent or divergent.

Example

Determine whether the integral

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}$$

is convergent or divergent. If the integral is convergent, find its value.

Solution: This integral is convergent with limit $3 + 3\sqrt[3]{2}$. Since it is comparable to a p-integral and $p < 1$ is a convergent region near vertical asymptotes. You are encouraged to work out the details of the limit for yourself.