Percents and Interest

Section 15.5

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Fall 2012

Working With Percentages

Suppose a quantity changes based on a percentage of its own value. Such *percent increases* or *percent decreases* are important calculations that arise in many applications including finance, economics, population growth and decay. Being able to calculate such changes efficiently is an important objective of this unit.

Problem: Suppose a quantity x...

1. increases by 8%

2. decreases by 5%

3. increases by 100%

4. decreases by 10%, then again decreases by 10%

Percent Change

Often times when looking at investments, we want to know the total percent change of an investment over time. The *percent change* (increase/decrease) can be expressed as the ratio of the differences between the value of the quantity initially and at present with the original value. In this case the formula is given by

$$\text{percent change} = \frac{\text{new value} - \text{original value}}{\text{original value}} \times 100\%.$$

In the case where this value is negative the formula denotes a percent decrease.

Example

Calculate the percent change of the value of an account that increases by 10% each year for two years.

Polling Question #25

If a stock rises by 3% in the first year and then loses 10% of its total value in the following year, what is the percentage change of the stock over the two years?

(a) 13.3% increase

(b) 7% decrease

(c) 13.3% decrease

(d) 7.3% decrease

Interest Terminology

Interest: Interest is the amount paid for lending or borrowing money, denoted by I. (Not to be confused with the interest rate, which is denoted by r.)

Principal: The principal is the amount of money borrowed or loaned, denoted by P or PV.

Future Value: The future value of an investment is the amount of money gained in interest and payments added to the initial value of the account. In other words, if A is the size of the account in the future, then

$$A = P + I.$$

Sometimes future value is denoted by FV instead of A.

Simple Interest: Interest calculated only on the principal amount is called simple.

Compound Interest: Interest calculated on not only the principal, but also on any accrued interest is called compound interest. The act of adding in the new interest is called compounding.

Adding Rates - Simple Interest

Example

A deposit of P dollars made today earns $r\%$ of its *initial* value every year for the next 5 years. How much money is in the account after 5 years? Money that grows in this way is called *simple interest*.

How would you change this to represent an arbitrary length of time t?

Definition (Formulas for Simple Interest)

A deposit of P dollars made today at a rate of interest r for t years produces interest according to

$$I = Prt.$$

The interest added to the original principal P gives

$$A = P + Prt = P(1 + rt).$$

The amount, A, is called the *future value of the deposit*.

Multiplying Rates - Compound Interest

While simple interest is normally used on investments or loans lasting less than a year, compound interest is used for longer periods. With compound interest, interest is charged (or paid) on interest as well as principal. The compound interest formula is really easy to obtain from the idea of simple interest though, as we will illustrate in the next example.

Example

Suppose you deposit $1000 for 2 years in an account paying 8% compounded quarterly (i.e. you add the accumulated interest in 4 times a year). If the interest for each quarter is calculated using simple interest, what is the value of the account after 2 years?

Deriving the Compound Interest Formula

Let's examine how this works for the first year. Recall the simple interest formula expresses the future value as $A = P(1 + rt)$ where r is the annual simple interest rate. In this case $r = 0.08$ and $t = 0.25$. This gives

$$\begin{aligned} A &= 1000\left(1 + \frac{0.08}{4}\right) = \$1020 \quad \text{(end of 1st quarter)} \\ &= 1020\left(1 + \frac{0.08}{4}\right) = \$1040 \quad \text{(end of 2nd quarter)} \\ &= 1040.40\left(1 + \frac{0.08}{4}\right) = \$1061.21 \quad \text{(end of 3rd quarter)} \\ &= 1061.21\left(1 + \frac{0.08}{4}\right) = \$1082.43 \quad \text{(end of 4th quarter)} \end{aligned}$$

which gives a total of \$1082.43 in the account at the end of the first year.

Example

How much would be in the account if we calculated the future value at the end of the year using simple interest?

Continuing on into the second year...

How would we adjust the formulas for the first year to describe both years?

$$\begin{aligned} A &= 1000\left(1+\frac{0.08}{4}\right)^4 \quad \text{(end of the first year)} \\ &= 1000\left(1+\frac{0.08}{4}\right)^4 \cdot \left(1+\frac{0.08}{4}\right)^4 \quad \text{(end of the second year)} \\ &= 1000\left(1+\frac{0.08}{4}\right)^{4\cdot 2} \end{aligned}$$

How much is in the account after t years if interest is compounded quarterly? What is the formula in general for any type of compounding?

$$A = P\left(1+\frac{0.08}{4}\right)^{4\cdot t}$$

where t is the number of years the money sits in the account earning interest. Here the 4 represents the number of times the interest is being compounded per year (i.e. quarterly), and 0.08 is the interest rate.

Compound Interest Formula

Based on the previous example, it seems like an appropriate formula would be

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where

$$\begin{aligned} n &= \text{number of compounding periods per year} \\ r &= \text{simple interest rate, as a decimal} \\ t &= \text{time, in years} \\ P &= \text{principal} \\ A &= \text{future value of the account} \end{aligned}$$

We have three ways to utilize this formula:

- By hand.
- Enter the equation into your calculator solver.
- Use your calculator's built in financial solver, i.e. TVM solver or Finance command.

Future Value of an Account

Example

Suppose you deposit \$12,000 into an account. What is the future value of the account if the money earns 5%

(a) simple interest

(b) compounded annually

(c) compounded monthly

What is the percent change of each account over that time?

Present Value of an Account

Example

Upon the birth of a new child a young couple wishes to establish an account to help pay for college when the child turns 18. How much money should the couple deposit on the day of the child's birth if they wish to save \$30,000 for college tuition? Assume their investment earns a rate of return of 8% compounded monthly. (What if they earned 8% simple interest instead?)

Application to Depreciation

Example

Car values depreciate quickly at first and then depending on the model more slowly over time. Suppose you purchase a new car for \$25,000 and it depreciates at a rate of 12% per year for the first 3 years. What will the value of the car be if you try to resell it in 3 years?

Example

Let's turn the previous example around a bit. Suppose this time that you buy a car for \$25,000 and five years later the car is worth \$13,550. What is the per year depreciation rate of the car?

Effective Rates

Sometimes, the rates stated in a problem, called the *nominal rate* (i.e. the rate in "name only") are not necessarily the rates that you pay. This is the case anytime interest is compounded more than once per year.

Definition (Effective Annual Yield)

The effective annual yield or effective rate is the simple interest rate that produces the same amount of money in an account at the end of one year as when the account is subjected to compound interest at a stated rate. The investment's effective annual yield, Y is given by

$$Y = \left(1 + \frac{r}{n}\right)^n - 1.$$

Comparing Investments and Loans

Example

Bank A is now lending money at 10% interest compounded annually. The rate at Bank B is 9.6% compounded monthly, and the rate at Bank C is 9.7% compounded quarterly. If you need to borrow money, at which bank will pay the least in interest charges?

In this case,

$$\begin{aligned} Y &= \left(1 + \frac{0.10}{1}\right)^{1} - 1 = 10\% \\ &= \left(1 + \frac{0.096}{12}\right)^{12} - 1 = 10.034\% \\ &= \left(1 + \frac{0.097}{4}\right)^{4} - 1 = 10.0586\% \end{aligned}$$

So Bank A is the best choice with the lowest APY even though it has the highest interest rate. This is often a source of confusion for consumers.

Polling Question #26

Suppose an account which is compounded annually doubles in value over the course of a year. Which of the following *must be false*!

(a) The percentage change of the account is 100%.

(b) The simple interest rate is the same as the compound interest rate.

(c) The APY of the account is higher than the nominal rate.

(d) The principal increases by 100%.