Trigonometric Integrals

Section 5.7

Dr. John Ehrke
Department of Mathematics

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Eliminating Powers From Trig Functions

Initially this lecture will not contain any new integral techniques. Our goal is to remind you of some important identities from trigonometry and reinforce the importance of substitutions and integration by parts we have already covered. It turns out that every anti derivative of the form

$$\int \cos^3 x\, dx, \qquad \int \sec^3 x\ \tan^3 x\, dx, \quad \text{and} \quad \int \cos^2 x\ \sin^3 x\ \tan^2 x\, dx$$

i.e, any product of integer powers of the six trigonometric functions can be solved in elementary form. The general technique involves applying reduction formulae and trig identities to reduce the powers. Some basic trigonometric identities you should recall:

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- (**Cosine Double Angle Identity**) $\cos(2\theta) = \cos^2\theta - \sin^2\theta$
- (**Sine Double Angle Identity**) $\sin(2\theta) = 2\sin\theta\cos\theta.$

New Formulas from Old

It turns out the previous three formulas are really the only ones we need as we can obtain all others needed from these three. In particular, we want to obtain a formula that allows us to reduce the powers of $\sin^n\theta$ and $\cos^n\theta$ for $n \geq 2$. We immediately turn to the cosine double angle identity, and rewrite it using the Pythagorean identity as

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta).$$

Solving this expression for $\cos^2\theta$ gives what is known as the *half-angle identity*,

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}.$$

In similar fashion we can obtain the half-angle identity for sine,

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The Basic Examples

We will start by considering the most important class of integrals for this topic–those of the form

$$\int \sin^n x \cos^m x \, dx, \qquad n, m = 0, 1, 2, \ldots.$$

These integrals show up in countless areas in mathematics including Fourier analysis.

The easiest case is when at least one of the exponents m or n is odd. We will begin with an example in this case.

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Evaluate $\int \sin^n x \cos x \, dx$ for any integer $n \geq 0$.

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Example

Evaluate $\int \sin^n x \cos x \, dx$ for any integer $n \geq 0$.

Solution: We use the substitution $u = \sin x$, in this case, $du = \cos x \, dx$, and so we have

$$\int \sin^n x \cos x \, dx = \int u^n \, dx = \frac{u^{n+1}}{n+1} + c = \frac{(\sin x)^{n+1}}{n+1} + c.$$

A Slightly Different Case

Example

Evaluate $\int \sin^3 x \; \cos^2 x \, dx$.

A Slightly Different Case

Example

Evaluate $\int \sin^3 x \ \cos^2 x \, dx$.

Solution: We are still in the easy case since $m = 3$ is odd. We use $\sin^2 x = 1 - \cos^2 x$ to eliminate the larger powers involved. So we have,

$$\int \sin^3 x \ \cos^2 x \, dx = \int (1 - \cos^2 x) \sin x \cos^2 x \, dx = \int (\cos^2 x - \cos^4 x) \sin x \, dx.$$

At this point we make the substitution $u = \cos x$ and $du = -\sin x \, dx$. Under this substitution we have

$$\int (u^2 - u^4)\,(-du) = -\frac{u^3}{3} + \frac{u^5}{5} + c.$$

Back substituting, we have

$$\int \sin^3 x \ \cos^2 x \, dx = -\frac{\cos^3 x}{3} + \frac{\cos^5 x}{5} + c.$$

No Cosine In Sight

Example

Evaluate $\int \sin^3 x \, dx$.

No Cosine In Sight

Example

Evaluate $\int \sin^3 x \, dx$.

Solution: We proceed as before and use $\sin^2 x = 1 - \cos^2 x$, to obtain

$$\begin{aligned} \int (1 - \cos^2 x) \; \sin x \, dx &= \int (1 - u^2) \, (-du) \\ &= -u + \frac{u^3}{3} + c \\ &= -\cos x + \frac{\cos^3 x}{3} + c \end{aligned}$$

The Harder Case

The harder case is when both of the powers are even. In this case you should try and apply the half-angle formulas.

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Evaluate $\int \cos^2 x \, dx$.

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Example

Evaluate $\int \cos^2 x\, dx$.

Solution: Applying the half-angle formula straight away gives

$$\int \cos^2 x\, dx = \int \frac{1+\cos(2x)}{2}\, dx = \frac{x}{2} + \frac{\sin(2x)}{2 \cdot 2} + c.$$

Another Harder Case

Example

Evaluate $\int \sin^2 x \, \cos^2 x \, dx$.

Another Harder Case

Example

Evaluate $\int \sin^2 x \ \cos^2 x \, dx$.

Solution: We do some scratch work off to the side to deal with the integrand. We have

$$\sin^2 x \cos^2 x = \left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right).$$

This is simply a difference of squares, so we have

$$\left(\frac{1-\cos(2x)}{2}\right)\left(\frac{1+\cos(2x)}{2}\right) = \frac{1-\cos^2(2x)}{4}.$$

We are still not done, but in light of the previous example, we use the half angle formula again and have

$$\frac{1-\cos^2(2x)}{4} = \frac{1}{4} - \frac{1+\cos(4x)}{4\cdot 2} = \frac{1}{8} - \frac{\cos(4x)}{8}.$$

From here the resulting integrand can easily be integrated.

An Alternative Method

Knowing the double angle identities can also be of use in this case.

Example

Evaluate $\int \sin^2 x \, \cos^2 x \, dx$ using a double angle identity.

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Example

Evaluate $\int \sin^2 x\ \cos^2 x\, dx$ using a double angle identity.

Solution: We recognize $\sin^2 x\ \cos^2 x = (\sin x \cos x)^2$. In this case we apply the double angle identity to get

$$(\sin x \cos x)^2 = \left(\frac{\sin(2x)}{2}\right)^2 = \frac{\sin^2(2x)}{4}.$$

We are still in the hard case at this point, but we can now apply the half-angle identity to obtain

$$\frac{\sin^2(2x)}{4} = \frac{1 - \cos(4x)}{8}.$$

The integration at this point gives

$$\int \sin^2 x\ \cos^2 x\, dx = \int \frac{1 - \cos(4x)}{8}\, dx = \frac{x}{8} - \frac{\sin(4x)}{32} + C.$$

A Word on Secant and Tangent

Recall the derivatives of tangent and secant are given by

$$\frac{d}{dx}\sec x = \sec x \tan x \qquad \text{and} \qquad \frac{d}{dx}\tan x = \sec^2 x.$$

These immediately give the following integral formulas:

$$\int \sec x \tan x \, dx = \sec x \qquad \text{and} \qquad \int sec^2 x \, dx = \tan x.$$

What about the integrals of just $\tan x$ and $\sec x$?

Example

Evaluate the integrals $\int \tan x \, dx$ and $\int \sec x \, dx$.

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Example

Evaluate the integrals $\int \tan x \, dx$ and $\int \sec x \, dx$.

Solution: First for $\tan x$ we have under the substitution $u = \cos x$,

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = \int -\frac{du}{u} = -\ln|\cos x| + c.$$

Solution continued...

Secondly, for the $\sec x$ we think about

$$\frac{d}{dx}(\sec x + \tan x) = \sec x \tan x + \sec^2 x = \sec x(\sec x + \tan x).$$

If we let $u = \sec x + \tan x$, this gives $u' = u \cdot \sec x$, or

$$\sec x = \frac{u'}{u} = \frac{d}{dx}\ln(u) = \frac{d}{dx}\ln(\sec x + \tan x).$$

Integrating both sides of this expression, we have

$$\int \sec x \, dx = \ln(\sec x + \tan x).$$

Similar techniques show that

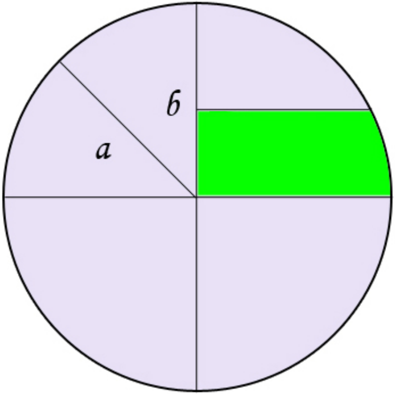
$$\int \csc x \, dx = -\ln|\csc x + \cot x| + c$$

and

$$\int \cot x \, dx = \ln|\sin x| + c.$$

Motivation for Trigonometric Substitutions

For the remainder of this lecture, we will consider several examples where making a trigonometric substitution can help simplify the process of integration. To motivate this method, consider the figure below.



What is the area of the green shaded region?

A New Integral

In this example, we would like to determine the area of the green shaded region. How might we go about doing this? Our first attempt might involve breaking the region up into vertical strips and integrating according to

$$\text{Area} = \int_0^a y\,dx$$

but this is potentially complicated since the function $y(x)$ is not a constant function. Rather than breaking the region up into vertical strips we might try horizontal strips. In this case our integral becomes,

$$\text{Area} = \int_0^b x\,dy = \int_0^b \sqrt{a^2 - y^2}\,dy.$$

The resulting integral is not one we've encountered so far, and based on our derivation of the integral and its associated area we might try a trigonometric approach.

Making the Trigonometric Substitution I

Example

Find the area of the shaded region by evaluating the integral $\int_0^b \sqrt{a^2 - y^2}\, dy$.

Solution: The picture suggests the substitution $y = a \sin\theta$. In this case, we have

$$\sqrt{a^2 - y^2} = \sqrt{a^2 - a^2 \sin^2\theta} = a\sqrt{1 - \sin^2\theta}$$

but we recognize $1 - \sin^2\theta = \cos^2\theta$, and so we have

$$\sqrt{a^2 - y^2} = a\cos\theta.$$

As a point of interest, on the previous slide, we had $x = \sqrt{a^2 - y^2}$, under this substitution $x = a\cos\theta$ only makes far too much sense.

Making the Trigonometric Substitution II

Going back to the integration, we have

$$\int_0^b \sqrt{a^2 - y^2}\, dy = \int_0^b (a\cos\theta)(a\cos\theta)\, d\theta$$

where $dy = a\cos\theta\, d\theta$. The resulting integral we handled in the previous lecture, and so we have

$$\text{Area} = a^2 \left(\frac{\theta}{2} + \frac{\sin(2\theta)}{4} \right) \Bigg|_0^b = a^2 \left(\frac{b}{2} + \frac{\sin(2b)}{4} \right).$$

Using Trigonometric Substitutions

Example

Evaluate the integral $\int \frac{dx}{x^2\sqrt{1+x^2}}$ using a trigonometric substitution.

Using Trigonometric Substitutions

Example

Evaluate the integral $\displaystyle\int \frac{dx}{x^2\sqrt{1+x^2}}$ using a trigonometric substitution.

Solution: Seeing the form $\sqrt{1+x^2}$ in the denominator suggests the substitution $x = \tan\theta$, since $\sqrt{1+x^2} = \sec\theta$ in this case. We have $dx = \sec^2\theta\, d\theta$, and substituting in we obtain

$$\int \frac{dx}{x^2\sqrt{1+x^2}} = \int \frac{\sec^2\theta}{(\tan^2\theta)(\sec\theta)}\, d\theta.$$

Generally, we will want to rewrite expressions such as this in terms of sines and cosines. In this case, after canceling the $\sec x$ in the denominator, we have

$$\int \frac{\cos^2\theta}{(\cos\theta)(\sin^2\theta)}\, d\theta = \int \frac{\cos\theta}{\sin^2\theta}\, d\theta.$$

We make one more substitution, $u = \sin\theta$, then $du = \cos\theta\, d\theta$ and we have

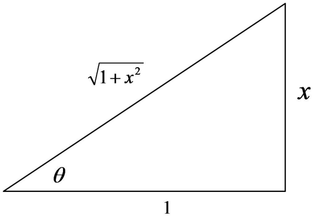
$$\int \frac{\cos\theta}{\sin^2\theta}\, d\theta = \int \frac{du}{u^2} = -\frac{1}{u} + c.$$

Let the Back Substituting Begin

In working backward now we have $u = \sin\theta$, so

$$-\frac{1}{u} + c = -\csc\theta + c,$$

but how do we make the second back substitution to recover the answer in terms of x. Recall we originally made the substitution $x = \tan\theta$. This determines the following triangle,



Based on this triangle, we see that $\csc\theta = 1/\sin\theta$ and so we have

$$\int \frac{dx}{x^2\sqrt{1+x^2}} = -\csc\theta + c = \frac{\sqrt{1+x^2}}{x} + c.$$

Please understand that what was going on here was simply evaluating the expression $-\csc(\theta) = -\csc(\arctan x)$ since $x = \tan\theta$ was our substitution.

What Substitution Do I Make?

There are essentially three ways in which the previous example can change and the all involve radicals of some type. The results of these substitutions are summarized in the table below.

Trigonometric Substitutions			
Radical Form	**Substitution**	***t*-Domain**	**Result**
$\sqrt{a^2 - x^2}$	$x = a\sin t$	$[-\pi/2, \pi/2]$	$\sqrt{a^2 - x^2} = a\cos t$
$\sqrt{a^2 + x^2}$	$x = a\tan t$	$[-\pi/2, \pi/2]$	$\sqrt{a^2 + x^2} = a\sec t$
$\sqrt{x^2 - a^2}$	$x = a\sec t$	$[0, \pi],\ x \neq \pi/2$	$\sqrt{x^2 - a^2} = \pm a\tan t$

Another Example

Example

Evaluate $\int \frac{dx}{\sqrt{x^2+4}}$ by using the appropriate trigonometric substitution.

Another Example

Example

Evaluate $\int \frac{dx}{\sqrt{x^2+4}}$ by using the appropriate trigonometric substitution.

Solution: Our substitution in this case is $x = 2\tan\theta$, then $dx = 2\sec^2\theta\, dt$ and $\sqrt{x^2+4} = 2\sec\theta$. Therefore,

$$\int \frac{dx}{\sqrt{x^2+4}} = \int \frac{2\sec^2\theta}{2\sec\theta}\, d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + c.$$

Back substituting, we observed that $\tan\theta = x/2$ and calculate $\sec(\arctan(x/2))$ to arrive at our answer

$$\int \sec\theta\, d\theta = \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + c.$$

Completing the Square

Example

Evaluate $\int \frac{dx}{\sqrt{x^2 + 2x + 5}}$ using a trigonometric substitution.

Completing the Square

Example

Evaluate $\int \frac{dx}{\sqrt{x^2+2x+5}}$ using a trigonometric substitution.

Solution: At first this example does not seem of the proper type to admit a trigonometric substitution, but completing the square in the denominator gives

$$\int \frac{dx}{\sqrt{x^2+2x+5}} = \int \frac{dx}{\sqrt{(x+1)^2+4}}.$$

Next, we make the direct substitution $u = x+1, du = dx$ and the result looks familiar

$$\int \frac{du}{\sqrt{u^2+4}}.$$

In this form we need only consult the previous problem for our answer.

A Secant Substitution

Example

Evaluate
$$\int_3^6 \frac{\sqrt{x^2-9}}{x}\,dx.$$

A Secant Substitution

Example

Evaluate

$$\int_3^6 \frac{\sqrt{x^2-9}}{x}\,dx.$$

Solution: We make the substitution $x = 3\sec\theta$, $dx = 3\tan\theta\sec\theta\,d\theta$. Upon substituting, we have $\sqrt{x^2-9} = 3\tan\theta$. When $x = 3$, we have $\sec\theta = 1$, and $\theta = 0$. When $x = 6$, we have $\sec\theta = 2$ and $\theta = \pi/3$. Therefore

$$\begin{aligned}
\int_3^6 \frac{\sqrt{x^2-9}}{x}\,dx &= \int_0^{\pi/3} \frac{3\tan\theta}{3\sec\theta}\cdot 3\tan\theta\sec\theta\,d\theta \\
&= 3\int_0^{\pi/3} \tan^2\theta\,d\theta \\
&= 3\int_0^{\pi/3} \sec^2\theta - 1\,d\theta \\
&= 3\left[\tan\theta - \theta\right]_0^{\pi/3} \\
&= 3\left(\sqrt{3} - \frac{\pi}{3}\right).
\end{aligned}$$