Infinite Sequences and Series

Section 8.1-8.2

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Zeno's Paradox

Achilles and the Tortoise

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-Aristotle

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Both of these paradoxes requires one to complete an infinite number of tasks, and so are thought be unachievable. We will illustrate this paradox by consider the infinite sum,

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = ??$$

What is the sum of this series?

Infinite Sequences

Definition (Infinite Sequence)

A sequence of real numbers is a real-valued function $a(n)$ whose domain is a subset of the non-negative integers. The numbers $a_n = a(n)$ are the terms of the sequence.

A simple example of an infinite sequence is given by

$$\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots, \frac{n}{n+1}, \ldots$$

where

$$a_1 = \frac{1}{1+1}, a_2 = \frac{2}{2+1}, a_3 = \frac{3}{3+1}, \ldots, a_n = \frac{n}{n+1}, \ldots$$

and a_n is called the *general term* of the sequence.

Definition (Convergent Sequence)

We say a sequence $a(n)$ converges to a limit L if and only if the limit of the general term of the sequence converges to L. That is, $a_n \to L$ as $n \to \infty$ if for each $\epsilon > 0$ there exists a positive integer N such that $|a_n - L| < \epsilon$ for all $n > N$.

What is the limit of the sequence above?

Two Types of Sequences

So I thought this was supposed to be a lecture about series. Why are we spending time on sequences? Well, it turns out that the question of the sum of an infinite series is intimately related to the question of the convergence of series.

Definition (Arithmetic Progression)

A sequence is called an arithmetic progression if it has the property that the difference between successive terms always has the same value. That is there is some number d, such that $a_{n+1} - a_n = d$ for all n.

For example, the sequence $1, 4, 7, 10, \ldots, 3n-2$ is arithmetic with $d = 3$ and general term $a_n = 3n - 2$. Because of this, this sequence *diverges* to $+\infty$ because $\lim_{n\to\infty} a_n = \infty$.

Definition (Geometric Progression)

A geometric progression is a sequence in which there is some number r, called the common ratio, with the property that

$$\frac{a_{n+1}}{a_n} = r, \qquad \text{for all } n.$$

Geometric Sequences

An example of a geometric progression is the sequence

$$\frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \ldots, \frac{3}{2^n}, \ldots \left(r = \frac{1}{2} \right).$$

It is clear that if $a_0, a_1, a_2, \ldots a_n$ is a geometric progression with common ratio r, then

$$a_n = a_0 r^n, \qquad n = 1, 2, \ldots.$$

This property of geometric progressions leads to the following result.

Geometric Progression Theorem

Suppose that $a_0, a_1, a_2, \ldots, a_n$ is a geometric sequence with common ratio r. Then one of the following must be true about $a(n)$:

1. If $-1 < r < 1$ is the common ratio of a geometric sequence, then the limit of the sequence is 0.
2. If $|r| > 1$ then the sequence will not converge.
3. If $r = 1$ then all the terms of the sequence are identical and the sequence converges to the value of a_0.
4. If $r = -1$, then the terms of the sequence alternate between $a_0, -a_0$ and so no limit exists.

Returning to Our Original Question

Definition (Partial Sum)

Let $a_1 + a_2 + \ldots + a_k + a_{k+1} + \ldots$ be an infinite series. In *sigma notation*,

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + \ldots + a_k + a_{k+1} + \ldots$$

The n^{th} partial sum, S_n is given by $S_n = a_1 + a_2 + a_3 + \ldots + a_n$.

Example

What are the partial sums for $n = 1, 2, 3, \ldots$ for the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots = ?$$

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Listing out the partial sums of this series, we obtain

$$S_1 = \frac{1}{2}, S_2 = \frac{3}{4}, S_3 = \frac{7}{8}, S_4 = \frac{15}{16}, \ldots, S_n = 1 - \frac{1}{2^n}.$$

Clearly the limit is $\lim_{n\to\infty} S_n = 1$, but what does this mean?

Convergence of Infinite Series

Given a set of numbers $\{a_1, a_2, a_3, \ldots\}$, the sum $a_1 + a_2 + a_3 = \sum_{i=1}^{\infty} a_i$ is called an *infinite series*. Its *sequence of partial sums*, $\{S_n\}$ has the terms:

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_n &= a_1 + a_2 + a_3 + \ldots + a_n = \sum_{i=1}^{n} a_i, \qquad n = 1, 2, 3, \ldots \end{aligned}$$

If the sequence of partial sums has a limit L, the infinite series *converges* to that limit, and we write

$$\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i = \lim_{n\to\infty} S_n = L.$$

If the sequence of partial sums diverges (i.e. the limit $L = \infty$, or does not exist), the infinite series also diverges.

Geometric Series

Definition (Geometric Series)

The geometric series is the infinite sum given by

$$\sum_{n=0}^{\infty} a_0 r^n = a_0 + a_0 r + a_0 r^2 + a_0 r^3 + \ldots + a_0 r^n + \ldots$$

Given this definition there are two questions which immediately arise.

(a) What are the values of r for which this series converges?

(b) What is the sum of the series when it is convergent?

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Solution: The partial sums of the geometric series are

$$s_1 = a_0, s_2 = a_0 + a_0 r, s_3 = a_0 + a_0 r + a_0 r^2, \ldots s_n = a_0(1 + r + r^2 + r^3 + \ldots + r^{n-1}).$$

If multiply each side of s_n by r we obtain

$$rs_n = a_0(r + r^2 + r^3 + \ldots r^n)$$

and subtracting we have $s_n - rs_n = a_0(1 - r^n)$. Solving for s_n we obtain the n^{th} partial sum of the geometric series

$$s_n = a_0\left(\frac{1 - r^n}{1 - r}\right).$$

Geometric Series Continued...

Sum of a Geometric Series

A geometric series

$$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + ar^3 + \ldots + ar^n + \ldots$$

converges if and only if $-1 < r < 1$ and diverges if $|r| \geq 1$. If the series converges, its sum is given by

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

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Proof: Because the general term of the sequence of partial sums for the geometric series is given by

$$s_n = a\left(\frac{1-r^n}{1-r}\right) = \frac{a}{1-r} - \frac{a}{1-r}r^n$$

and $-1 < r < 1$ then

$$\lim_{n\to\infty} S_n = \frac{a}{1-r}$$

giving the result. It is one of the most important objectives of this unit that you are able to use this result to solve geometric series.

Properties of Convergent Series

1. Suppose $\sum a_k$ converges to A and let c be a real number. The series $\sum ca_k$ converges and
$$\sum ca_k = c\sum a_k = cA.$$

2. Suppose $\sum a_k$ converges to A and $\sum b_k$ converges to B. The series $\sum(a_k \pm b_k)$ converges and
$$\sum (a_k \pm b_k) = \sum a_k \pm \sum b_k = A \pm B.$$

3. Whether a series converges does not depend on a finite number of terms added to or removed from the series. Specifically, if M is a positive integer, then
$$\sum_{k=1}^{\infty} a_k \quad \text{and} \quad \sum_{k=M}^{\infty} a_k$$
both converge or both diverge. However, the value of the series *would change*.

4. (Re-indexing) Changing the index of a sum does not change the convergence of the sum. For example,
$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} a_{k+1}.$$
Notice subtracting one from the index results in adding one to the term of the series.

Using the Geometric Series Formula

Example

Evaluate the infinite series

$$S = \sum_{k=1}^{\infty} \left[5 \left(\frac{2}{3} \right)^k - \frac{2^{k-1}}{7^k} \right].$$

Using the Geometric Series Formula

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Evaluate the infinite series

$$S = \sum_{k=1}^{\infty} \left[5\left(\frac{2}{3}\right)^k - \frac{2^{k-1}}{7^k} \right].$$

Solution: Recall that the sum of a geometric series is given by

$$\sum_{n=0}^{\infty} r^n = \frac{a}{1-r}.$$

Using this we have

$$\sum_{k=1}^{\infty} 5\left(\frac{2}{3}\right)^k = \left[\frac{5(2/3)}{1-\frac{2}{3}}\right] = 10$$

and

$$\sum_{k=1}^{\infty} \frac{2^{k-1}}{7^k} = \frac{1/7}{1-2/7} = \frac{1}{5}.$$

both series converge, and so their combined value is $S = 10 - \frac{1}{5} = \frac{49}{5}$.

Infinite Series and Notation

A series $S = \sum_{k=1}^{\infty} a_k$ has partial sums, S_n, given by $S_n = \dfrac{n}{2n-1}$.

1. Is $\sum_{k=1}^{\infty} a_k$ convergent or divergent? If it is convergent, what is the sum?
2. What is $\lim_{k->\infty} a_k$?
3. What is the value of a_9?
4. What is the value of the sum, $\sum_{n=1}^{10} a_k$?

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4. What is the value of the sum, $\sum_{n=1}^{10} a_k$?

Solution: Clearly this is convergent since $\lim_{n\to\infty} S_n = 1/2$. So the sum, $S = 1/2$. Since the sum converges, we must have $\lim_{k\to\infty} a_k = 0$, or else the sum wouldn't converge. The value of $a_9 = S_9 - S_8$ which is $a_9 = 9/17 - 8/15 = -1/255$. Finally,

$$\sum_{n=1}^{10} a_k = S_{10} = \frac{10}{19}.$$

A Telescoping Sum In Two Ways

Example

Consider the infinite series

$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)}.$$

Find the first four terms of the sequence of partial sums. Find an expression for S_n and make a conjecture about the value of series. Prove your conjecture.

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Example

Consider the infinite series
$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)}.$$
Find the first four terms of the sequence of partial sums. Find an expression for S_n and make a conjecture about the value of series. Prove your conjecture.

Solution: The sequence of partial sums can be evaluated as follows:
$$S_1 = 1/2, S_2 = 1/2 + 1/6 = 2/3, S_3 = 1/2 + 1/6 + 1/12 = 3/4, \ldots S_n = n/n+1.$$
Because the $\lim\limits_{n\to\infty} n/n+1 = 1$, we conclude that $\lim_{n\to\infty} S_n = 1$. Using partial fractions, the sequence of partial sums is
$$S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left(\frac{1}{i} - \frac{1}{i+1} \right).$$
Writing out this sum, we see that
$$S_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1} = 1 - \frac{1}{n+1}.$$

An Unusual Application

Example

Express the repeating decimal 5.2323232323... as an improper fraction.

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Express the repeating decimal 5.2323232323... as an improper fraction.

Solution: To express the repeating decimal as a fraction we recognize the decimal as being the same as

$$5.232323... = 5 + \frac{23}{10^2} + \frac{23}{10^4} + \frac{23}{10^6} + \dots$$

Looking at the series created, if we factor out the beginning term, we have,

$$5.232323... = 5 + \frac{23}{10^2}\left[1 + \frac{1}{10^2} + \frac{1}{10^4} + \dots\right]$$

which is a geometric series. Using the sum formula for the geometric series, we have

$$5.232323... = 5 + \frac{23}{10^2} \cdot \frac{1}{1-(1/100)} = 5 + \frac{23}{99} = \frac{518}{99}.$$