Second Order Equations, Three Cases

Sections 3.1-3.4

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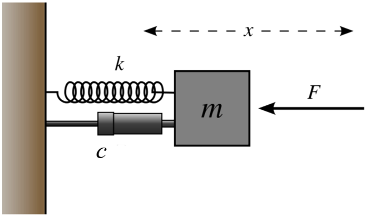
Spring Mass Systems

Second order ODEs most frequently arise in mechanical spring mass systems. We will consider such models going forward since they represent the most important class of models for these equations.



- k, spring constant (Hooke's Law)
- m, mass of the weight
- F, external force (for a homogeneous model, $F = 0$)
- c, damping constant (determine the effect the dashpot has on resisting the motion of the spring)

What is the second order differential equation which describes this model?

The Spring Mass Model

Consider the figure of the spring-mass model on the previous slide. We have attached a mass m to the spring. This weight stretches the spring until it reaches an equilibrium at $x = x_0$. At this point there are two forces acting on the mass. There is the force of gravity, mg, and there is the restoring force of the spring, which we denote by $R(x)$ since it depends on the distance x that the spring is stretched. Because we are at equilibrium at $x = x_0$ these two forces must be acting in direct opposition to one another. That means

$$R(x_0) + mg = 0 \tag{1}$$

In addition to the gravity and restoring force, when the spring is stretched there is a damping force D, which is the resistance to the motion of the weight due to the medium or in this case a dashpot. The major dependence in this case is on the velocity, so we write $D = D(v)$. According to Newton's second law,

$$\begin{aligned} ma &= \text{total force acting on the weight} \\ &= R(x) + mg + D(v) + F(t) \end{aligned} \tag{2}$$

where $F(t)$ is an external force applied to put the system into motion, or keep the system in motion.

Putting it all together...

For many springs, the restoring force is proportional to the displacement. This result is called *Hooke's law*. It says that

$$R(x) = -kx \tag{3}$$

We use the minus sign and assume $k > 0$, because the restoring force is acting to decrease the displacement. Assuming for the moment there is no external force and that the weight is at spring-mass equilibrium where $x = x_0$ and $x' = x'' = 0$, then the damping force is $D = 0$ and we have

$$0 = R(x_0) + mg = -kx_0 + mg \quad \text{or} \quad mg = kx_0.$$

The damping force, $D(v)$ always acts against the velocity, so we write it as,

$$D(v) = -cv \tag{4}$$

Substituting $mg = kx_0$ into equation (2) with $a = x''$ and $v = x'$ gives

$$mx'' = -k(x - x_0) - cx' + F(t) \tag{5}$$

This motivates use to introduce the variable $y = x - x_0$, upon which we obtain the spring mass equation

$$my'' = -ky - cy' + F(t) \quad \text{or more commonly} \quad my'' + cy' + ky = F(t). \tag{6}$$

Harmonic Motion Equation

If we divide equation (6) by the mass $m \neq 0$, and make the identifications $p = c/2m$, $\omega_0 = \sqrt{k/m}$, $f(t) = F(t)/m$, and $x = y$, we obtain the equation

$$x'' + 2px' + \omega_0^2 x = f(t) \tag{7}$$

where $p \geq 0$ and $\omega_0 > 0$ are constants. We refer to equation (7) as the equation for harmonic motion. It models a great many phenomena including the vibrating spring system just mentioned as well as the general behavior of an RLC circuit.

- The constant p is called the *damping constant*.
- The constant ω_0 is called the *natural frequency*.
- The function $f(t)$ is called the *forcing term*.

We will begin by analyzing the simplest form of this equation in the absence of damping ($p = 0$), and then study the three cases that can result when damping is allowed.

Simple Harmonic Motion

In the special case where there is no damping ($p = 0$) the spring mass model, equation (7) simplifies to

$$x'' + \omega_0^2 x = 0. \tag{8}$$

The characteristic equation is $r^2 + \omega_0^2 = 0$ which gives characteristic roots $r = \pm i\omega_0$. This means a fundamental set of solutions for (8) are given by

$$x_1 = e^{i\omega_0 t} \quad \text{and} \quad x_2 = e^{-i\omega_0 t},$$

so the general solution is a linear combination of x_1 and x_2 which we express as

$$x(t) = c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t} \quad \Longrightarrow \quad x(t) = a\cos(\omega_0 t) + b\sin(\omega_0 t). \tag{9}$$

There is perhaps more that we should say about this last equation. In particular, where did the imaginary part go?

Complex Solutions

In studying the case with complex roots as in the last slide, we run into the problem that our solutions are complex. We will handle this case by appealing to the following theorem.

Theorem

If $u(t) + iv(t)$ is a complex valued solution to the second order differential equation $my'' + cy' + ky = 0$, where m, c, k are real, nonzero constants, then in addition to this solution, $u(t)$ and $v(t)$ are both solutions of this equation as well.

Proof: If $y(t) = u(t) + iv(t)$ solves the constant coefficient second order equation (with $c/m = a$, and $k/m = b$) given by $y'' + ay' + by = 0$, then

$$(u'' + iv'') + a(u' + iv') + b(u + iv) = 0.$$

This implies that

$$(u'' + au' + bu) + i(v'' + av' + bv) = 0,$$

but this says that $u(t)$ and $v(t)$ are solutions.

Oscillations

So in our solution of the simple harmonic motion equation, we obtain

$$x(t) = c_1 e^{i\omega_0 t} + c_2 e^{-i\omega_0 t} = (c_1 \cos \omega_0 t - c_2 \sin \omega_0 t) + i(c_2 \cos \omega_0 t - c_1 \sin \omega_0 t).$$

By the previous theorem we may choose either the real or imaginary part of this equation as a solution (remember, there are many different fundamental sets of solutions). Thus, the solution to the simple harmonic motion equation is a pure oscillation of the form (with $a = c_1, b = -c_2$),

$$x(t) = a \cos(\omega_0 t) + b \sin(\omega_0 t). \tag{10}$$

In the case of a vibrating spring without friction, we see that the mass on the spring oscillates up and down with natural frequency $\omega_0 = \sqrt{k/m}$. Borrowing a trick from trigonometry and our knowledge of polar coordinates, we can write equation (10) as a single oscillation,

$$x(t) = A \cos(\omega_0 t - \phi)$$

where A is the *amplitude* of the oscillation, ω is the *circular frequency*, ϕ is the *phase lag*, and $P = 2\pi/\omega_0 = 2\pi\sqrt{m/k}$ is the period.

Another solution form...

We could stop here and be perfectly happy in our analysis, but it is beneficial to understand that many engineers actually prefer to deal with the solution in the general form

$$x(t) = c_1 e^{(a+bi)t} + c_2 e^{(a-bi)t}.$$

The only problem with this form is a priori we have no guarantee that a solution in this form is real. Under what conditions is this a real solution?

Given a complex expression if upon changing i to −i in the expression, and nothing changes, then the original expression must be real.

Using this bit of information, we examine the above equation and see that upon changing i to $-i$, we obtain

$$x(t) = \overline{c_1} e^{(a-bi)t} + \overline{c_2} e^{(a+bi)t}$$

which is the same as the original equation if and only if $\overline{c_1} = c_2$, i.e. they are complex conjugates of one another. In this case, we write,

$$x(t) = (c + id)e^{(a+ib)t} + (c - id)e^{(a-ib)t}.$$

Are our two forms in agreement?

The Damped Equation

We return to our harmonic motion equation developed in the last lecture, but will instead consider solutions to this equation in the case where $p > 0$. Recall the equation in this case is given by

$$x'' + 2px' + \omega_0^2 x = 0 \tag{11}$$

and has characteristic equation

$$r^2 + 2pr + \omega_0^2 = 0 \tag{12}$$

with roots,

$$r_1 = -p - \sqrt{p^2 - \omega_0^2} \quad \text{and} \quad r_2 = -p + \sqrt{p^2 - \omega_0^2}. \tag{13}$$

In the case where the roots are imaginary, we emphasize the form of the roots by explicitly pulling out the imaginary term as follows:

$$r_1 = -p - i\sqrt{\omega_0^2 - p^2} \quad \text{and} \quad r_2 = -p + i\sqrt{\omega_0^2 - p^2}. \tag{14}$$

The Three Cases

Considering the discriminant of the roots in equation (13) we will have three cases to consider:

(i) $p < \omega_0$ This is the *underdamped* case. The roots are distinct complex numbers. Hence, the general solution is given by

$$x(t) = e^{-pt}\left[c_1 \cos \omega t + c_2 \sin \omega t\right]$$

where $\omega = \sqrt{\omega_0^2 - p^2}$.

(ii) $p > \omega_0$ This is the *overdamped* case. The roots are distinct and real. Further, since $\sqrt{p^2 - \omega_0^2} < \sqrt{c^2} = c$, we have $r_1 < r_2 < 0$. The general solution is

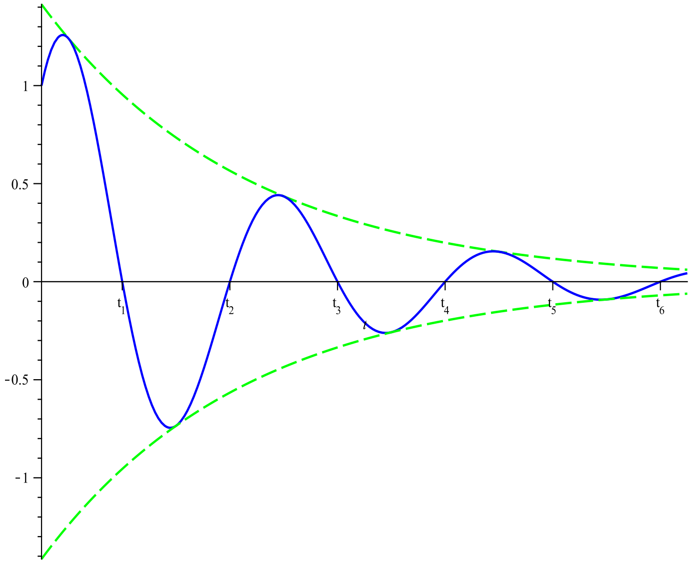
$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

(iii) $p = \omega_0$ This is the *critically damped* case, and in this case, the root is a double root, $r = -p$, and so as we will show on the next slide, the general solution is

$$x(t) = c_1 e^{-pt} + c_2 t e^{-pt}.$$

In each of these cases the solution decays to zero as $t \to \infty$ due to the exponential term in the solution, and the fact that $p > 0$.

Analyzing an Oscillation



The Repeated Root Case

Let $r = -a$ be a repeated root. In this case, the characteristic equation is given by

$$(r+a)^2 = 0 \quad \Longrightarrow \quad r^2 + 2ar + a^2 = 0.$$

If we assume that one solution to the equation

$$x'' + 2ax' + a^2 x = 0$$

is of the form $x_1(t) = e^{-at}$, then what is our second? Well, let's suppose that if a second solution exists it is of the form $x_2(t) = x_1 \cdot u$ for some unknown function $u(t)$. Substituting in x_2 to our equation, and combining like terms, we obtain

$$e^{-at}u'' = 0 \quad \Longrightarrow \quad u'' = 0.$$

The most general solution of this equation is $u(t) = c_1 + c_2 t$, but we choose $c_1 = 0, c_2 = 1$. Hence the second of our fundamental set of solutions is

$$x_2(t) = te^{-at}.$$

Example 1: The Overdamped Case

Example

Obtain a general, and particular solution for the second order ODE

$$y'' + 4y' + 3y = 0$$

with initial conditions $y(0) = 1, y'(0) = 0$. What do the initial conditions suggest about the motion of the mass? What is the general behavior of solutions in this case?

Solution: The characteristic equation is given by $r^2 + 4r + 3 = 0$, which leads to roots, $r = -3, r = -1$. Our general solution is given by

$$y(t) = c_1 e^{-3t} + c_2 e^{-t}.$$

Solving for the initial conditions yields, $c_1 = -1/2, c_2 = 3/2$, so the solution to the IVP is:

$$y(t) = -\frac{1}{2}e^{-3t} + \frac{3}{2}e^{-t}.$$

Example 2: The Underdamped Case

Example

Obtain a general, and particular solution for the second order ODE

$$y'' + 4y' + 5y = 0$$

with initial conditions $y(0) = 1, y'(0) = 0$. Write your solutions in terms of a single oscillation. Describe the behavior of solutions in this case.

Solution: The characteristic equation is given by $r^2 + 4r + 5 = 0$ with complex conjugate roots, $r = 2 \pm i$. The general solution is given by

$$y(t) = e^{-2t}\left(c_1 \cos t + c_2 \sin t\right).$$

Applying the initial conditions gives $c_1 = 1, c_2 = 2$. Written as a single oscillation we obtain

$$y(t) = \sqrt{5}e^{-2t}\cos(t - \phi)$$

where $\phi = \arctan 2 \approx 1.107$.

Example 3: The Critically Damped Case

Example

Obtain a general, and particular solution for the second order ODE

$$y'' + 6y' + 9 = 0$$

with initial conditions $y(0) = 1, y'(0) = 0$.

Solution: The characteristic equation is given by $r^2 + 6r + 9 = 0$ with repeated root $r = -3$. This gives a general solution of

$$x(t) = c_1 e^{-3t} + c_2 te{-}3t.$$

Solving for the initial conditions gives $c_1 = 1$ and $c_2 = 3$. In this case, the particular solution is

$$y(t) = e^{-3t} + 3te^{-3t}.$$

The Three Cases Theorem

Theorem

Let y_1 and y_2 be two linearly independent solutions of the homogeneous equation

$$y'' + Ay' + By = 0$$

with constant coefficients A and B. If $y(t)$ is any solution for this equation then $y(t) = c_1y_1 + c_2y_2$ where the y_1 and y_2 depend on solutions of the characteristic equation $r^2 + Ar + B = 0$. If this equation has,

(a) two real solutions $r_1 \neq r_2$, then $y(t) = c_1e^{r_1t} + c_2e^{r_2t}$.

(b) a complex conjugate pair $r = a \pm bi$, then $y(t) = e^{at}(c_1 \cos bt + c_2 \sin bt)$.

(c) a repeated real root r, then $y(t) = c_1e^{rt} + c_2te^{rt}$