Calculating Volumes

Section 6.2-6.3

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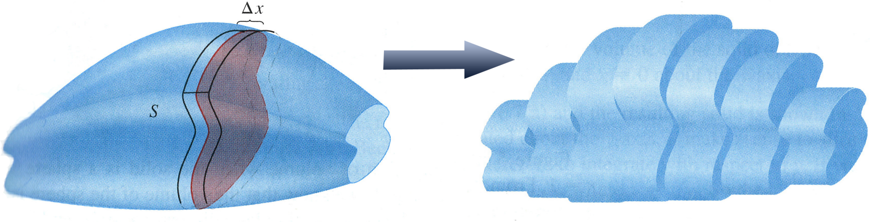
What We Mean By Slicing

Intuitively, finding volumes of solids is not that different from finding area. We already have nice formulas for figures such as cylinders, prisms, and spheres, but how are these formulas derived, and better yet, how do we handle a solid S like the one below?



If we let $A(x)$ denote the area of a slice and Δx denote the width of the slice, then we can make a rudimentary estimate of the volume of a single slice as $\Delta V \approx A(x)\Delta x \implies dV = A(x)\,dx$. In an intuitive way we make an analogy to Riemann sums and can say that if we make n slices starting from $x = a$ to $x = b$, then the volume of the solid S is given by $V = \sum_{i=1}^{n} A_i(x)\Delta x$ and as we let $n \to \infty$ (i.e. make more, thinner slices), we obtain

$$V(x) = \int_a^b A(x)\,dx.$$

Volumes by Slicing, An Example

Example

Derive the formula for the volume of a right pyramid whose height is h and whose base is a square with sides of length a. The formula from geometry states that $V = \frac{1}{3}$(area of the base)(height). Can we reproduce this result using calculus techniques?

Solution: This is the most generic type of slicing problem, but in some ways is the most difficult type since it requires a careful extraction of the formula for the cross-sectional areas $A(x)$. Let the pyramid be centered about $(0,0)$ with the y-axis along the height and the x-axis supporting the base. At any y in the interval $[0,h]$ on the y-axis the cross section perpendicular to the y axis is a square. If s denotes the length of a side of this square, then by similar triangles,

$$\frac{s/2}{a/2} = \frac{h-y}{h}$$

and since we after an expression for s in terms of y we have $s = \frac{a}{h}(h-y)$. Thus the area $A(y)$ of the cross section at y is given by

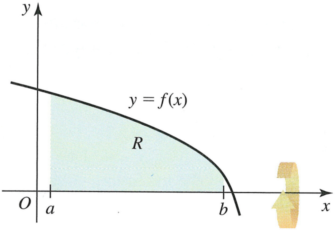
$$A(y) = s^2 = \frac{a^2}{h^2}(h-y)^2.$$

By our previous result then the volume is given by

$$V = \int_0^h A(y)\,dy = \int_0^h \frac{a^2}{h^2}(h-y)^2\,dy = \frac{1}{3}a^2h.$$

Solids of Revolution, Disk Method

The unfortunate part of the previous problem is that we are really limited by the nature of the function $A(x)$. In particular, we must know something about the geometry of the objects produced via slicing. This means we will consider very special types of solids called *solids of revolution*. A solid of revolution is obtained by taking a function in the xy-plane (usually in a single quadrant) and revolving that graph around the x or y axis to obtain a solid figure.



Considering the solid of revolution formed by revolving the region R about the x-axis, we note that the cross sectional areas are circular discs with radius $f(x)$. Therefore, the cross section at the point x, where $a \leq x \leq b$, has area

$$A(x) = \pi(\text{radius})^2 = \pi f(x)^2.$$

By our general slicing method, the volume of the solid of revolution is given by

$$V = \int_a^b A(x)\, dx = \int_a^b \pi f(x)^2\, dx.$$

Because each slice is a circular disk, this method is often called the *disk method*.

A Disk Method Example

Example

Find the volume of a ball of radius a centered at $(a, 0)$.

Solution: Recall, the rectangular equation for this circle is given by

$$(x-a)^2 + y^2 = a^2.$$

Solving for y^2 we obtain

$$y^2 = a^2 - (x-a)^2 = 2ax - x^2.$$

Plugging this into our formula for the volume we obtain

$$V = \int_0^{2a} \pi(2ax - x^2)\, dx = \frac{4}{3}\pi a^3.$$

This is of course the familiar formula for the sphere of radius a. In general, we have the formula for the volume of the portion of the sphere given by

$$V(x) = \pi\left(ax^2 - \frac{x^3}{3}\right).$$

Rotating the Other Way

Example

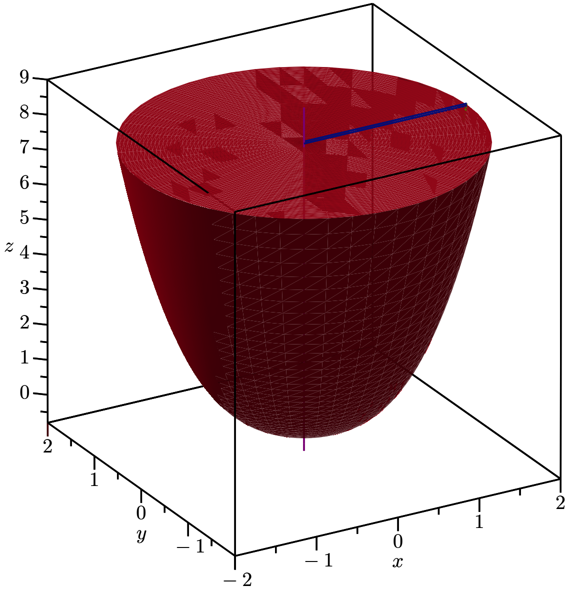
Find the volume of the solid obtained by rotating the region bounded by $y = x^3$, $y = 8$, and $x = 0$ about the y-axis.

Solution: This solid has a slightly different orientation than those before since it is revolved around the y-axis. It makes sense here to slice perpendicular to the y axis and then integrate with respect to y, as if we were rotating the image down to the x axis. Given that our function is $y = x^3$, our slice covers a horizontal distance of x units, where $x = \sqrt[3]{y}$. The area of each cross section (once again a circle) is given by

$$A(y) = \pi x^2 = \pi(\sqrt[3]{y})^2 = \pi y^{2/3}.$$

Since the solid lies between $x = 0$ and $x = 8$, its volume is

$$V = \int_0^8 A(y)\,dy = \int_0^8 \pi y^{2/3}\,dy = \frac{96}{5}\pi.$$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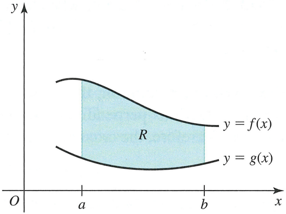
The Washer Method

A slight variation on the disk method allows us to compute the volume of more exotic solids of revolution. Consider the region R pictured below.



Once again we apply the slicing method, but in this case the cross-sectional area is not a disk, but rather a *washer* with an outer radius of $R = f(x)$ and a hole with a radius of $r = g(x)$, where $a \leq x \leq b$. The area of the cross section is the area of the entire disk minus the area of the hole, or

$$A(x) = \pi(R^2 - r^2) = \pi(f(x)^2 - g(x)^2).$$

The volume is then calculated in the usual way

$$V(x) = \int_a^b \pi(f(x)^2 - g(x)^2)\, dx.$$

A Washer Method Example

Example

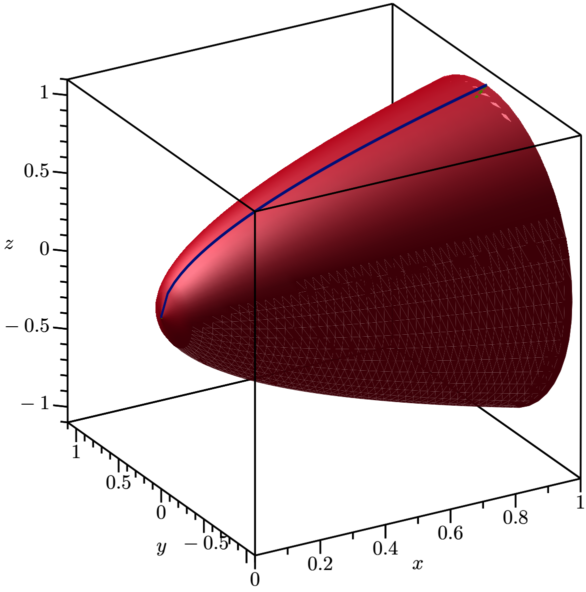
Suppose the region R is bounded by the graphs of $f(x) = \sqrt{x}$ and $g(x) = x^2$ between $x = 0$ and $x = 1$. What is the volume of the solid that results from revolving R about the x-axis?

Solution: In this case, $f(x) = \sqrt{x} \geq g(x) = x^2$ on the interval from $0 \leq x \leq 1$. The area of a typical cross section at the point x is

$$A(x) = \pi((\sqrt{x})^2 - (x^2)^2) = \pi(x - x^4).$$

Therefore the volume of the solid is

$$V = \int_0^1 \pi(x - x^4)\, dx = \pi \left(\frac{x^2}{2} - \frac{x^5}{5} \right)\Bigg|_0^1 = \frac{3\pi}{10}.$$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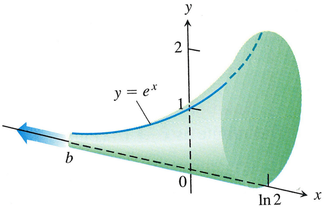
1

A Peculiar Example

The cross sections of the solid horn (pictured below) perpendicular to the x-axis are circular disks with diameters reaching from the x-axis to the curve $y = e^x$, $-\infty < x \leq \ln 2$. Find the volume of the horn.



Solution: The area of a typical cross section is

$$A(x) = \pi \left(\frac{1}{2}y\right)^2 = \frac{\pi}{4}e^{2x}.$$

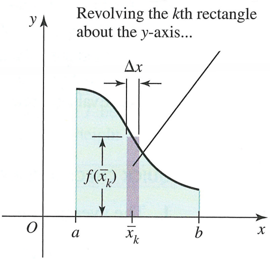
We define the volume of the horn to be the limit as $b \to -\infty$ of the volume of the solid pictured above give by

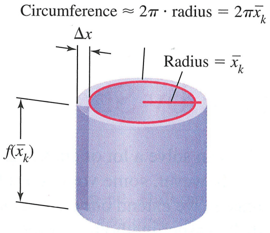
$$V = \int_b^{\ln 2} \frac{\pi}{4}e^{2x}\,dx = \frac{\pi}{8}\left(e^{\ln 4} - e^{2b}\right) = \frac{\pi}{8}\left(4 - e^{2b}\right).$$

As $b \to -\infty$, $e^{2b} \to 0$, and so $V \to \pi/2$. The volume of the horn is finite despite having infinite length!

The Method of Shells

Consider a region R (pictured below) that is revolved around the y-axis generating a solid of revolution. Now observe the rectangle on the k^{th} subinterval with height $f(\bar{x}_k)$ and width Δx. As it revolves around the y-axis, this rectangle sweeps out a thin *cylindrical shell*.





When the k^{th} shell is unwrapped it approximates a thin rectangular slab. The length of the slab is the circumference of a circle with radius $\bar{x}_k$, which is $2\pi\bar{x}_k$. This means the volume of the k^{th} shell is approximately

$$\Delta V = 2\pi\bar{x}_k \cdot f(\bar{x}_k) \cdot \Delta x = 2\pi\bar{x}_k f(\bar{x}_k)\Delta x.$$

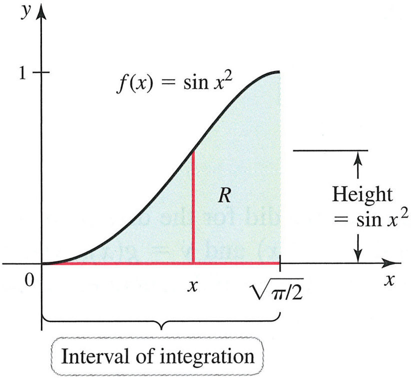
Summing the volumes of the n cylindrical shells gives an approximation to the volume of the entire solid which gets better as $\Delta x \to 0$ giving

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} 2\pi\bar{x}_k f(\bar{x}_k)\Delta x = \int_a^b 2\pi x \cdot f(x)\, dx.$$

Sine Bowl

Example

Let R be the region bounded by the graph of $y = \sin x^2$, the x-axis, and the vertical line $x = \sqrt{\pi/2}$. Find the volume of the solid generated when R is revolved about the y-axis.



Solution continued...

Revolving R about the y-axis produces a bowl shaped region. The radius of a typical shell is x and its height is $y = \sin x^2$. Therefore the volume by the shell method is

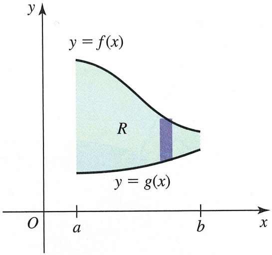
$$V = \int_0^{\sqrt{\pi/2}} 2\pi x \cdot \sin x^2 \, dx.$$

Now we make the change of variables $u = x^2$ which means that $du = 2x\, dx$. The lower limit $x = 0$ becomes $u = 0$ and the upper limit $x = \sqrt{\pi/2}$ becomes $u = \pi/2$. The volume of the solid is

$$V = \int_0^{\sqrt{\pi/2}} 2\pi x \cdot \sin x^2 \, dx = \pi \int_0^{\pi/2} \sin u \, du = \pi.$$

The General Shells Method

To generalize the method of shells we proceed in much the same way we did with the disk method progressing to the washer method. Suppose that the region R is bounded by two curves, $y = f(x)$ and $y = g(x)$, where $f(x) \geq g(x)$ on $[a, b]$. What is the volume of the solid generated when R is revolved about the y-axis?



The only real change made here is the height of the shell, which translates to a change in the volume of the shell. In this case the exact volume is given by the definite integral

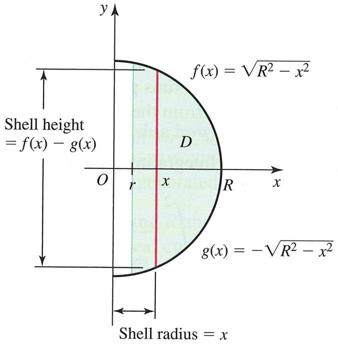
$$V = \int_a^b 2\pi x \cdot (f(x) - g(x))\, dx.$$

Volume of a Drilled Sphere

In this example, we return to the problem of finding the volume of a sphere, but in this case, we will drill a hole through the middle of the sphere and then find its volume.

Example

A cylindrical hole with radius r is drilled symmetrically through the center of a sphere of radius R, where $r \leq R$. What is the volume of the resulting solid?



Solution continued...

The y-axis is chosen to coincide with the axis of the cylindrical hole. The radius of a typical shell is x and its height is $f(x) - g(x) = 2\sqrt{R^2 - x^2}$. Therefore the volume of the solid of revolution is given by

$$
\begin{aligned}
V &= \int_r^R 2\pi x \cdot (2\sqrt{R^2 - x^2})\, dx \\
&= -2\pi \int_{R^2-r^2}^0 \sqrt{u}\, du \\
&= 2\pi \left(\frac{2}{3} u^{3/2} \right) \Bigg|_0^{R^2-r^2} \\
&= \frac{4\pi}{3} \left(R^2 - r^2 \right)^{3/2}
\end{aligned}
$$

In the case where $r = R$ (the extreme case where the hole is the whole sphere), then we have zero volume. In the case that $r = 0$ (no hole in the sphere), our calculation gives the correct volume of the sphere, $\frac{4}{3}\pi R^3$.

Which Method to Use?

Example

Consider the region R bounded by the graphs of $f(x) = 2x - x^2$ and $g(x) = x$ on the interval $[0, 1]$. Which method should/will you use to solve for the volume of the solid formed by revolving R about the x-axis.

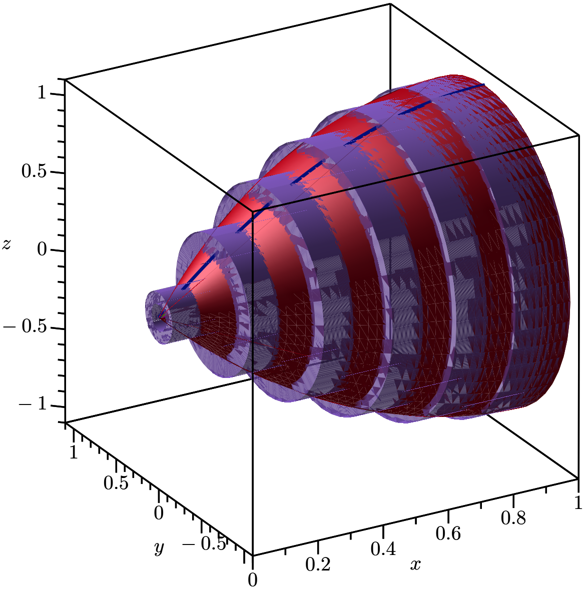
Solution: In the case of the washer method the resulting integral is

$$V = \int_0^1 \pi \left[(2x - x^2)^2 - x^2\right]\, dx = \pi \int_0^1 x^4 - 4x^3 + 3x^2\, dx = \frac{\pi}{5}.$$

If the shell method is used the bounding functions are given by $x = 1 - \sqrt{1 - y}$ and $x = y$. In this case the volume is given by

$$V = \int_0^1 2\pi y \left(y - (1 - \sqrt{1 - y})\right)\, dy = \frac{\pi}{5}.$$

This integral is decidedly more difficult to evaluate so the washer method is the preferred method here, but both give identical results.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Revolving Around An Arbitrary Axis

Example

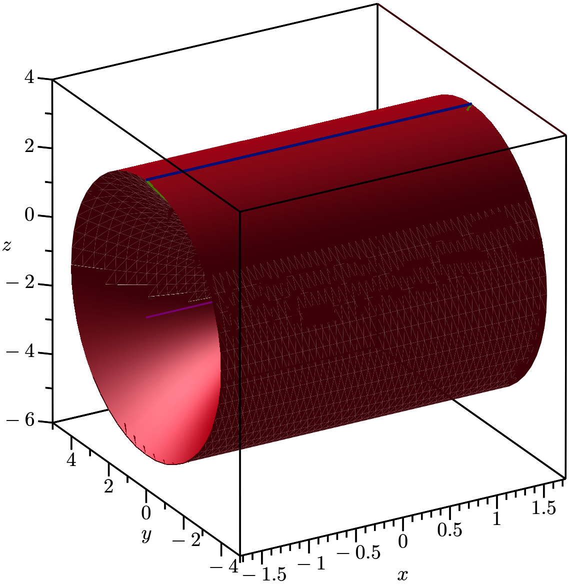
Consider the region R bounded by the graphs of $y = x^2$ and $y = 3$. What is the volume of the solid of revolution formed when R is rotated about the line $y = -1$?

Solution: It will be best to use cylindrical shells to solve this problem using horizontal strips. We have $x = \sqrt{y}$. A vertical strip of width dy sweeps out a cylindrical shell whose axis is along the line $y = -1$. Such a shell has radius $y - (-1) = y + 1$ and its height is $2x = 2\sqrt{y}$. So the volume is

$$dV = 2\pi(y+1)(2\sqrt{y})\,dy = 4\pi\sqrt{y}(y+1)\,dy$$

and so the integral giving the volume is

$$V = 4\pi \int_0^3 \sqrt{y}(y+1)\,dy = \frac{112\sqrt{3}\pi}{5} \approx 121.9.$$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