Substitution Rule

Sections 5.5

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Making a Change of Variables

The most common technique for evaluating integrals (both definite and indefinite) is making a substitution. To this end, we will consider an example to start with and then make some general observations.

Example

Evaluate the indefinite integral $\int 2x \cdot (x^2 - 5)^3 \, dx$.

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The most common technique for evaluating integrals (both definite and indefinite) is making a substitution. To this end, we will consider an example to start with and then make some general observations.

Example

Evaluate the indefinite integral $\int 2x \cdot (x^2 - 5)^3 \, dx$.

Solution: Let $u(x) = x^2 - 5$, then calculating the differential, $du = 2x \, dx$. Making this substitution gives

$$\int 2x \cdot (x^2 - 5)^3 \, dx = \int u^3 \, du.$$

Substitution trades one integral for another and this new integral is elementary. We evaluate this new integral

$$\int u^3 \, du = \frac{u^4}{4} + c.$$

We back substitute in terms of x, and obtain $\dfrac{u^4}{4} + c = \dfrac{(x^2 - 5)^4}{4} + c.$

Adjusting Your Substitution

You should not expect every substitution to go that neatly. Usually you will need to adjust a constant as in the following example.

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Evaluate the indefinite integral $\int \frac{2x}{\sqrt{3x^2 - 1}}\, dx$.

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Evaluate the indefinite integral $\int \frac{2x}{\sqrt{3x^2 - 1}}\, dx$.

Solution: Let $u(x) = 3x^2 - 1$, then $du = 6x\, dx$, but upon examination of the integrand we need $2x\, dx$ not $6x\, dx$. If we go ahead and make this substitution then we have multiplied by 3 times too much. To offset this we divide by 3 as well. In this case we obtain,

$$\begin{aligned} \int \frac{2x}{\sqrt{3x^2 - 1}}\, dx &= \frac{1}{3} \cdot \int \frac{du}{\sqrt{u}} \\ &= \frac{2}{3}\sqrt{u} + c \\ &= \frac{2}{3}\sqrt{3x^2 - 1} + c \end{aligned}$$

Inverse Substitutions

Sometimes you think a particular substitution should work, only to find a variable or two have been unaccounted. Not all hope is lost in this case, as we will see in the next example.

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Evaluate the indefinite integral $\displaystyle\int \frac{x}{\sqrt{2x+3}}\,dx$.

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Evaluate the indefinite integral $\int \frac{x}{\sqrt{2x+3}}\,dx$.

Solution: We start as usual by letting $u = \sqrt{2x+3}$, then $du = dx/\sqrt{2x+3}$. Substitution gives

$$\int \frac{x}{\sqrt{2x+3}}\,dx = \int x \frac{dx}{\sqrt{2x+3}} = \int x\,du.$$

What about the remaining x? The trick is to write x in terms of u, as follows

$$u = \sqrt{2x+3} \Longrightarrow u^2 = 2x+3 \Longrightarrow \frac{u^2-3}{2} = x.$$

Solution continued...

Making this new inverted substitution, we have

$$\begin{aligned}\int x\,du &= \int \frac{u^2-3}{2}\,du\\ &= \frac{u^3}{6} - \frac{3u}{2} + c\\ &= \frac{(2x+3)^{3/2}}{6} - \frac{3\sqrt{2x+3}}{2} + c.\end{aligned}$$

Often times when a variable is left out like this, the answer was to make your initial substitution in terms of x and dx rather than u and du. In this case, we would have had $dx = u\,du$ and $x = (u^2-3)/2$, and would have gotten

$$\int \frac{x}{\sqrt{2x+3}}\,dx = \int \frac{u^2-3}{2u}\,u\,du = \int \frac{u^2-3}{2}\,du$$

as before.

Multiple Substitutions

In some cases multiple substitutions are possible to achieve the same answer. Keep this in mind when trying to simplify your work.

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Evaluate the definite integral $\int \frac{dx}{1+\sqrt{x}}$.

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Evaluate the definite integral $\int \frac{dx}{1+\sqrt{x}}$.

Solution: Let $u = \sqrt{x}$, then using inverse substitution we have $x = u^2$ and $dx = 2u\, du$. Substituting for x and dx gives

$$\int \frac{dx}{1+\sqrt{x}} = \int \frac{2u}{1+u}\, du.$$

A second substitution is needed, but we are making progress as the second integral was simpler than the first. If $v = 1+u$, then $dv = du$ and $u = v-1$, so

$$\int \frac{2u}{1+u}\, du = \int \frac{2(v-1)}{v}\, dv = 2\int \left(1 - \frac{1}{v}\right)\, dv = 2(v - \ln \mid v \mid) + c.$$

Solution continued...

Finally, we substitute back for x,

$$2(v - \ln \mid v \mid) + c = 2(1 + u - \ln \mid 1 + u \mid) + c = 2 + 2\sqrt{x} - 2\ln \mid 1 + \sqrt{x} \mid + c.$$

It turns out a better first substitution would work in avoiding having to make two substitutions. We let $\sqrt{x} = u - 1$, then $x = u^2 - 2u + 1 = (u-1)^2$ and $dx = 2(u-1)\, du$. Substituting for x and dx,

$$\int \frac{2(u-1)}{u}\, du = 2\int du - 2\frac{du}{u} = 2u - 2\ln u + c.$$

Back substituting for x, we have

$$2u - 2\ln u + c = 2(1 + \sqrt{x}) - 2\ln(1 + \sqrt{x}) + c.$$

This is probably not the best form to leave our answer in since it includes a constant 2 which we can lump in with the constant c. Making this change we have

$$2\sqrt{x} - 2\ln(1 + \sqrt{x}) + d.$$

What about definite integrals?

In the case of definite integral substitutions we must take the limits into account as well. This next example will illustrate the process.

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Evaluate the definite integral $\int_{\pi/2}^{\pi} \sin(y) \cdot e^{\cos(y)}\, dy$.

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Example

Evaluate the definite integral $\int_{\pi/2}^{\pi} \sin(y) \cdot e^{\cos(y)}\, dy$.

Solution: We make the substitution $u = \cos y$, then $du = -\sin y$. Under these substitutions, we evaluate $u(y)$ at each of the limits of integration and obtain $u(\pi/2) = \cos(\pi/2) = 0$, and $u(\pi) = \cos \pi = -1$. In this case, we have

$$\int_{\pi/2}^{\pi} \sin y\, e^{\cos y}\, dy = -\int_{0}^{-1} e^{u}\, du = \int_{-1}^{0} e^{u}\, du = e^{0} - e^{-1} = 1 - \frac{1}{e}.$$

Notice we did not need to back substitute before evaluating the integral.