Integration by Parts

Section 5.6

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Inverse of the Product Rule

In some sense, the substitution rule covered in the previous lecture is the reverse of the chain rule for derivatives. In a similar sense, integration by parts primary function is undoing the product rule for derivatives. Recall, the product rule for derivatives says that, for differentiable functions u and v,

$$(u \cdot v)' = u \cdot v' + v \cdot u'.$$

From the anti derivative point of view, the product rule says that uv is an antiderivative of $uv' + u'v$. In symbols,

$$\int (u(x) \cdot v'(x) + v(x) \cdot u'(x))\, dx = u(x) \cdot v(x).$$

Equivalently,

$$\int u(x) \cdot v'(x)\, dx = u(x) \cdot v(x) - \int v(x) \cdot u'(x)\, dx.$$

This is the formula for integration by parts and this technique always works, albeit slowly sometimes.

The Steps Involved

What you hope for when using the integration by parts technique is to trade a difficult integral problem for a slightly simpler problem. Picking $u(x), v'(x)$ wisely is often the key to making this process go smoothly. We outline the steps involved with the following example, which on the surface seems intractable.

Example

Use integration by parts to evaluate $\displaystyle\int \ln x \, dx$.

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Example

Use integration by parts to evaluate $\int \ln x \, dx$.

Solution: If we set $u(x) = \ln x$, then $dv = 1dx$, so $du = 1/x \, dx$ and $v(x) = x$. Using the integration by parts formula, this gives

$$\int \ln x \, dx = x \cdot \ln |x| - \int 1 \, dx = x \cdot \ln x - x + c.$$

Notice, that the choice of $u(x) = 1$ would cause $dv = \ln x \, dx$ and make the process redundant.

Taking it Further

Let's see if we can expand this idea to a point where we can use it to define a family of anti derivatives.

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Use the results of the previous example to evaluate $\int (\ln x)^2\, dx$.

Solution: In this case, we are going to let $u(x) = (\ln x)^2$ and $dv = 1\, dx$ and so $v(x) = x$ and $du = 2 \ln x \cdot (1/x)$. We'll throw this at the formula and see what we get.

$$\int (\ln x)^2\, dx = x \cdot (\ln x)^2 - \int 2 \ln x \cdot \frac{1}{x} \cdot x\, dx = x \cdot (\ln x)^2 - \int 2 \ln x\, dx.$$

But this new integral can be resolved from our previous example. In this case, we have

$$\int (\ln x)^2\, dx = x \cdot (\ln x)^2 - 2(x \cdot \ln \mid x \mid - x) + c.$$

One More Time, Obtaining a Reduction Formula

Example

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Use the results of the previous examples to evaluate $\int (\ln x)^n \, dx$ for $n = 0, 1, 2, \ldots$.

Solution: Let $u(x) = (\ln x)^n$, and $dv = 1\,dx$ as before. Then $v(x) = x$ and $dv = n(\ln x)^{n-1}(1/x)$. Putting this all together, we obtain

$$x \cdot (x \ln x)^n - n \int (\ln x)^{n-1} \cdot \frac{1}{x} \cdot x \, dx.$$

If we use the notation $F_n(x) = \int (\ln x)^n \, dx$. Then $F_n(x) = x(\ln x)^n - nF_{n-1}(x)$. This completely describes the process hinted at by the previous examples. For example,

$$\begin{aligned} F_0(x) &= \int (\ln x)^0 \, dx = x + c \\ F_1(x) &= x \ln x - F_0(x) = x \ln x - x + c \\ F_2(x) &= x(\ln x)^2 - 2F_1(x) = x(\ln x)^2 - 2(x \ln x - x) + c \end{aligned}$$

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Solution: A choice needs to be made here as to which function we choose for $u(x)$. Notice that only one of the functions is made simpler under differentiation, so we choose $u(x) = x^n$ and $dv = e^x \, dx$. This gives $du = nx^{n-1} \, dx$ and $v(x) = e^x$. Integrating by parts, we obtain

$$\int x^n e^x \, dx = x^n e^x - \int nx^{n-1} e^x \, dx.$$

We summarize this recurrence with $G_n(x) = \int x^n e^x \, dx$, then $G_n(x) = x^n e^x - nG_{n-1}(x)$. So we have,

$$\begin{aligned} G_0(x) &= e^x + c \\ G_1(x) &= xe^x - G_0(x) = xe^x - e^x + c \\ G_2(x) &= x^2 e^x - 2(xe^x - e^x) + c \end{aligned}$$

Open Class Example

Example

Evaluate $\int x\sqrt{x+1}\,dx$ using any analytic method of your choice.

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Evaluate $\int x\sqrt{x+1}\,dx$ using any analytic method of your choice.

Solution: Let $u = x$ and $dv = (x+1)^{1/2}$. Then $du = dx$ and $v = \frac{2}{3}(x+1)^{3/2}$. Putting this all together we have

$$\begin{aligned}\int x\sqrt{x+1}\,dx &= x \cdot \frac{2}{3}(x+1)^{3/2} - \int \frac{2}{3}(x+1)^{3/2}\,dx \\ &= \frac{2}{3}x(x+1)^{3/2} - \frac{2}{3}\cdot\frac{2}{5}(x+1)^{5/2} + C \\ &= \frac{2}{3}x(x+1)^{3/2} - \frac{4}{15}(x+1)^{5/2} + C\end{aligned}$$

Integrating Twice and Then Solving

We will come back to this example later in the semester and simplify it somewhat, but for now we consider an example which on the surface seems to go in circles.

Example

Use integration by parts to evaluate $\int e^x \sin x \, dx$.

Integrating Twice and Then Solving

We will come back to this example later in the semester and simplify it somewhat, but for now we consider an example which on the surface seems to go in circles.

Example

Use integration by parts to evaluate $\int e^x \sin x \, dx$.

Solution: If we set $u(x) = \sin x$, then $dv = e^x \, dx$, $du = \cos x \, dx$, $v = e^x$ and

$$I = \int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx.$$

We haven't made much progress, so let's try integration by parts again on the new integral. If $u(x) = \cos x$, then $dv = e^x \, dx$, $du = -\sin x \, dx$ and $v = e^x$, so

$$I = e^x \sin x - \int e^x \cos x \, dx = e^x \sin x - \left(e^x \cos x + \int e^x \sin x \, dx \right).$$

The original integral I, has reappeared. We can solve the last equation for I, and obtain

$$I = \frac{1}{2} \left(e^x \sin x - e^x \cos x \right) + c.$$

Handling Definite Integrals

Definite Integration by Parts

Let $u(x)$ and $v(x)$ be differentiable functions, and suppose $u'(x)$ and $v'(x)$ are continuous on $[a, b]$, then

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\big|_a^b - \int_a^b u'(x)v(x)\,dx.$$

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Example

Evaluate $\int_0^\pi x\cos(2x)\,dx$.

Solution: We choose $u(x) = x$, and $dv = \cos(2x)\,dx$, so that $du = 1\,dx$ and $v(x) = \sin(2x)/2$ and we have

$$\frac{x}{2}\sin(2x)\Big|_0^\pi - \frac{1}{2}\int_0^\pi \sin(2x)\,dx.$$

This gives

$$0 - \frac{1}{2}\left(-\frac{1}{2}\cos(2x)\right)\Big|_0^\pi = 0 - \frac{1}{2}\left(-\frac{1}{2}+\frac{1}{2}\right) = 0.$$

Tabular Integration by Parts Example

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Evaluate $\int x^3 \sin x \, dx$ using integration by parts.

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Solution: We will begin by making a table containing the derivatives of $p(x) = x^3$ (what we would have called u previously) and $f(x) = \sin x$ (what we would have called dv previously). The result is the following table.

$\pm$	d/dx	$\int dx$
+	x^3	$\sin x$
-	$3x^2$	$-\cos x$
+	$6x$	$-\sin x$
-	6	$\cos x$
+	0	$\sin x$

Now simply pair the 1st entry of the first column with the 2nd entry of the second column, until further pairing results in a zero pair. The result in this case is

$$\begin{aligned}&(+)(x^3)(-\cos x) + (-)(3x^2)(-\sin x) + (+)(6x)(\cos x) + (-)(6)(\sin x)\\ &\qquad = -x^3\cos x + 3x^2\sin x + 6x\cos x - 6\sin x + C\end{aligned}$$

Open Class Example

Example

Evaluate $\int 2x^3 e^{3x}\, dx$ using tabular integration.

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Solution: We apply integration by parts again rather than do three integration by parts calculations.

$\pm$	d/dx	$\int\, dx$
+	$2x^3$	e^{3x}
-	$6x$	$e^{3x}/3$
+	6	$e^{3x}/9$
-	0	$e^{3x}/27$

Our answer then is given by

$$\frac{2}{3}x^3e^{3x} - \frac{2}{3}xe^{3x} + \frac{2}{9}e^{3x} + C.$$

Not everything is integrable

Some functions are not integrable. This means there is no closed form answer which describes their antiderivative. The function $f(x) = e^{x^2}$ is a classic example of this, but sometimes these functions when paired with other integrable functions can yield results.

Example

For a given non-negative integer n, does $x^n \cdot e^{x^2}$ have an elementary anti derivative?

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For a given non-negative integer n, does $x^n \cdot e^{x^2}$ have an elementary anti derivative?

Solution: For $n = 0$, no in that case, the integrand is e^{x^2} which is not integrable. For $n = 1$, the answer is yes, since the substitution $u = x^2$ produces the anti derivative $\exp(x^2)/2$. We might guess then, that the answer is yes for odd n and no for even n.

Checking our Guess

Our guess is correct. To see why we let $u = x^{n-1}$ and $dv = x \cdot \exp(x^2)\,dx$ leads to

$$\int x^n e^{x^2}\,dx = x^{n-1}\frac{e^{x^2}}{2} - \frac{n-1}{2}\int x^{n-2}e^{x^2}\,dx.$$

We observe two things:

- The left-hand integrand has an elementary anti derivative if and only if the one on the right does.
- Applying the reduction repeatedly knocks down the power of x by 2 each time. We'll eventually reach either 0 or 1, depending on whether n is odd or even. These facts combined with what we know for $n = 0$ and $n = 1$, show our conjecture to be correct.