Principle of Mathematical Induction

Strong Induction

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Unstacking Problem

Example

Suppose we wish to analyze strategies for winning the following game. You begin with a stack of n boxes. Then you make a sequence of moves. In each move, you divide one stack of boxes into two non-empty stacks. The game ends when you have n stacks, each containing a single box. You earn points for each move; in particular, if you divide on stack of height $a + b$ into two stacks with heights a and b, then you score $a \cdot b$ for that move. Your overall score is the sum of the points that you earn for each move. What strategy should you use to maximize your total score?

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As an example, suppose that we begin with a stack of $n = 10$ boxes. Then the game might proceed as follows:

Stack Heights										Score
<u>10</u>										
5	<u>5</u>									25 points
<u>5</u>	3	2								6
<u>4</u>	3	2	1							4
2	<u>3</u>	2	1	2						4
<u>2</u>	2	2	1	2	1					2
1	<u>2</u>	2	1	2	1	1				1
1	1	<u>2</u>	1	2	1	1	1			1
1	1	1	1	<u>2</u>	1	1	1	1		1
1	1	1	1	1	1	1	1	1	1	1
						Total Score			=	45 points

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1	1	$\underline{2}$	1	2	1	1	1			1
1	1	1	1	$\underline{2}$	1	1	1	1		1
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Can you find a better strategy?

We'll analyze the unstacking game using a variant of induction called *strong induction*. Strong induction and weak induction are used for the exact same purpose: proving that a proposition $P(n)$ is true for all $n \in \mathbb{N}$.

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Theorem (Principle of Strong Induction)

Let $P(n)$ be a proposition. If

- *$P(0)$ is true, and*
- *for all $n \in \mathbb{N}$, $P(0), P(1), P(2), \ldots, P(n)$ imply $P(n+1)$*

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The only change from weak induction is that strong induction allows you to assume more in the inductive step of the proof. In a srong induction argument, you may assume $P(0), P(1), \ldots, P(n)$ are all true to aid you in proving that $P(n+1)$ is true.

Strategy is Irrelevant

Theorem

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The proof will proceed via strong induction. Let $P(n)$ be the proposition that every way of unstacking n blocks gives a score of $n(n-1)/2$. Then if $n = 1$, there is only one block, and hence no unstacking moves are possible which gives 0 points, and so the total score for the game is indeed

$$\frac{1(1-1)}{2} = 0.$$

Therefore, $P(1)$ is true.

Now, assume that $P(k)$ is true for all $1 \leq k \leq n-1$. We will show that $P(n)$ is true. Consider a stack of n boxes. The first unstacking will produce two substacks with sizes k and $n-k$ for some $1 \leq k \leq n-1$. Now the total score of the game is the sum of points for the first move plus points obtained by unstacking the two resulting substacks (which are handled by the inductive hypothesis). Thus the total score is given by

$$\begin{aligned}\text{score} &= k(n-k) + \frac{k(k-1)}{2} + \frac{(n-k)(n-k-1)}{2} \\ &= \frac{2nk - 2k^2 + k^2 - k + n^2 - nk - n - nk + k^2 + k}{2} \\ &= \frac{n(n-1)}{2}\end{aligned}$$

Why was strong induction the right choice?

- Knowing information about how to unstack a stack of 9 boxes for example, does not provide any information about how to unstack a stack of 10 boxes in general, since there are more initial moves than splitting the stack into 9 and 1. In general, because the idea of unstacking is arbitrary in terms of the number of cups chosen for the first two stacks, it was necessary to assume the truth of all previous attempts.
- The technique of strong induction is logically equivalent to weak induction, in that strong induction essentially assumes for an initialized loop, you've run the first k passes with no change in the truth value of your proposition. Does this imply that pass $k+1$ will hold? Well, yes, if your proposition is a loop invariant.
- In the abstract, it is good enough to assume it worked for the k^{th} pass (i.e. weak induction), but in reality as a proof technique, sometimes it is easier to apply strong induction since it gives you more information upon which to build a proof.

Theorem

Let P be a polygon in the plane. To triangulate P is to draw diagonals through the interior of the polygon so that (1) the diagonals do not cross each other, and (2) all resulting regions are triangles. An exterior triangle is formed in the case where two of the three sides of one of the triangular regions uses the exterior of the original polygon. We claim that if a polygon with four or more sides is triangulated, then at least two of the triangles formed are exterior.

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Let n denote the number of sides of the polygon. We will prove the above claim via strong induction on n. Since this result only makes sense for a polygon with four sides and above, the basis case is $n = 4$. In this case, a triangulation of a quadrilateral consists of a single diagonal drawn in one of two possible orientations. Either way, the triangles formed must be exterior. Now assume the result to be true for $n = 4, 5, \ldots, k$. Let P be any triangulated polygon with $k + 1$ sides. Let d be one of the diagonals of P in its triangulation. The diagonal d separates P into two polygons A and B where A and B are themselves triangulated polygons having fewer sides than P. It is possible that one or both of A and B are triangles themselves, or neither. So we consider cases:

- If A and B are not triangles. Then by the inductive hypothesis each will be guaranteed to have 2 exterior triangles, though it is not necessary they both be exterior triangles of P since one of these exterior triangles might have the diagonal d as one of its exterior sides. Nonetheless, the other exterior triangle of A can't also use d and so A contributes one exterior triangle. The same is true of B.
- If A and B are both triangles, then clearly A and B are both exterior triangles of P by construction.
- Any combination of the above two arguments works for the case in which one of A or B is a triangle, but not the other.

Theorem

Every positive integer $n \geq 2$ can be written as a product of primes.

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Proof.

We proceed via strong induction. Let $P(n)$ be the statement that n can be written as a product of primes. The base case is when $n = 2$, 2 itself is prime so $P(2)$ holds. We show that $P(2) \wedge P(3) \wedge \ldots \wedge P(n) \Rightarrow P(n+1)$ is true. Assume that $P(2), P(3), \ldots, P(n)$ are true. Consider $n+1$. If $n+1$ is a prime number, we are done. Otherwise, $n+1$ is a composite number, and so it has factors, u and v with $2 \leq u, v < n+1$ such that $u \cdot v = n+1$. By the inductive hypothesis,

$$u = \prod_i p_i, \qquad v = \prod_j p_j$$

for primes p_i and p_j. Therefore,

$$n+1 = \prod_i p_i \cdot \prod_j p_j$$

and so $P(n+1)$ is true. By the principle of strong induction, $P(n)$ is true for all $n \geq 2$. $\square$

Problem: Given an unlimited supply of 3 cent and 5 cent postage stamps, what postages are possible?

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Solution: Let's first try to guess the answer and then try to prove it. The table below show values of all possible combinations of 3 and 5 cent stamps where the column heading is the number of 5 cent stamps and the row heading is the number of 3 cent stamps.

	0	1	2	3	4	5	...
0	0	5	10	15	20	25	...
1	3	8	13	18	23	...	
2	6	11	16	21	...		
3	9	14	19	24	...		
4	12	17	22	...			
5	15	20	...				
⋮	⋮	⋮					

Looking at the table, a resonable guess is that the possible postages are 0, 3, 5, and 6 cents and every value of 8 or more cents. Let's try to prove this last part using strong induction.

Theorem

For all $n \geq 8$ *it is possible to produce* n *cents of postage from 3 ¢ and 5 ¢ stamps.*

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Proof.

We will proceed via strong induction on n. Let $P(n)$ be proposition that "if $n \geq 8$, then n ¢ stamp postage can be produced using 3 ¢ and 5 ¢ stamps. $P(8)$ is clearly true since we produce 8 ¢ postage using one 3 ¢ one 5 ¢stamp. Assuming the strong induction hypothesis that we know how to produce k ¢ of postage for all values of k between 8 and n, we can let $k = n - 2$ and produce k ¢ of postage; then add a 3 ¢ stamp to get $n + 1$ cents. But we have to be careful because if $n + 1$ is 8, 9, or 10, this doesn't work. Those cases are easily resolved in a trio of base case arguments where the case for 8 ¢ is given above, 9 ¢is a multiple of 3, and 10 ¢ is a multiple of 5.

Now for $n + 1 > 10$ we have $n \geq 10$, so $k = n - 2 \geq 8$ and by strong induction we may assume we can produce exactly $n - 2$ cents of postage. Adding 3 ¢ stamp to this gives $n + 1$ cents of postage using only 3 ¢ stamps and 5 ¢ stamps, i.e. $P(n + 1)$ is true. □