Calculus II

Improper Integration

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Improper Integrals

Integrals can commit two types of impropriety:

Type 1 The interval of integration may be infinite. This is technically illegal, because the formal definition of definite integral relies on partitions that are finite.

Type 2 The integrand may be unbounded on the interval of integration. This is also illegal, because the integrand is not defined at an endpoint of the interval of integration.

Example 1

For each of the following integrals, determine the type of impropriety that occurs.

(a)
$$\int_1^\infty \frac{dx}{x^2}$$

(b)
$$\int_1^\infty \frac{dx}{x}$$

(c)
$$\int_{-\infty}^\infty \frac{dx}{1+x^2}$$

(d)
$$\int_0^\infty e^{-x^2}\,dx$$

(e)
$$\int_0^1 \frac{dx}{x^2}$$

(f)
$$\int_0^1 \frac{dx}{\sqrt{x}}$$

(g)
$$\int_0^\infty \frac{dx}{\sqrt{x}+x^2}$$

It will pay to keep these integrals in the back of your mind. They illustrate most of what can go right–and wrong–with improper integration.

Type I: Infinite Intervals

Improper Integrals of Type I

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leq b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

(c) If both of the above integrals are convergent (i.e. their limits exist), then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx.$$

We note that any value $x = a$ can be used to split the integral.

Example 2

Determine whether the integral

$$\int_1^\infty \frac{dx}{x^2}$$

converges or diverges. If it converges find its limit.

Example 3

Determine whether the integral

$$\int_1^\infty \frac{dx}{x}$$

converges or diverges. If it converges find its limit.

Example 4

Determine whether the integral

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$$

converges or diverges. If it converges find its limit.

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Type 2: Discontinuous Integrands

Improper Integrals of Type 2

(a) If f is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

(c) If f has a discontinuity at $c\in(a,b)$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^a f(x)\,dx$$

and has limits according to those types above.

Example 5

Determine whether the integral

$$\int_0^1 \frac{dx}{x^2}$$

converges or diverges. If it converges find its limit.

Example 6

Determine whether the integral

$$\int_0^1 \frac{dx}{\sqrt{x}}$$

converges or diverges. If it converges find its limit.

Example 7

Determine whether the integral

$$\int_0^\infty \frac{dx}{\sqrt{x}+x^2}$$

converges or diverges. If it converges find its limit.

Comparison Test for Improper Integrals

Comparison Test

Suppose that f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$.

(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.

(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Example 8

Does

$$I = \int_1^\infty \frac{dx}{x+1}$$

converge or diverge? Why?

Example 9

Does

$$I = \int_3^\infty \frac{2x^2}{x^4 + 2x + \cos x}\, dx$$

converge or diverge? Why?

Absolute Comparison Test for Improper Integrals

Absolute Convergence

Let f and g be continuous functions such that, for all $x \geq a$,

$$0 \leq |f(x)| \leq g(x).$$

Suppose that $\int_a^\infty g(x)\,dx$ converges. Then $\int_a^\infty f(x)\,dx$ also converges, and

$$\left|\int_a^\infty f(x)\,dx\right| \leq \int_a^\infty g(x)\,dx.$$

Example 10

Does the integral

$$I = \int_1^\infty \frac{\sin x}{x^2}\, dx$$

converge or diverge? Why?