LaTeX Writing Assignment

Binary Arithmetic: From Leibniz to von Neumann *

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Instructions

In this project you will explore the foundations of the binary number system. You should begin by reading through this document. This is a significant course requirement. (Each reflection paper constitutes 10% of your course grade.) The historical presentation is often more verbal than modern textbook formulations, which carries the advantage that no technical knowledge is assumed a priori. On the other hand, you must carefully read the historical author's words and be prepared to experiment with your own calculations to verify or amplify the historical source. Here are some specific instructions.

1. The final work should be a written paper, composed in LaTeX, in which you address all issues raised in the project. Use complete sentences along with modern notations, equations, etc... to support your claims, but do not simply list the questions posed in this paper and answer them. Your work should possess narrative qualities, including an introduction, conclusion, and a well organized flow.
2. Reflect upon and discuss any connections from the historical source with present day techniques that you may have encountered. How does the historical source differ from textbook presentation?
3. You should seek additional historical or mathematical information about the principle parties of the project, and provide insights to the ideas presented in this project that are a result of your findings.

The Era of Leibniz

Gottfried Wilhelm Leibniz (1646–1716) is often described as the last universalist, having contributed to virtually all fields of scholarly interest of his time, including law, history, theology, politics, engineering, geology, physics, and perhaps most importantly, philosophy, mathematics and logic [1, 9, 11]. The young Leibniz began to teach himself Latin at the age of 8, and Greek a few years later, in order to read classics not written in his native language, German. Later in life, he wrote:

> Before I reached the school-class in which logic was taught, I was deep into the historians and poets, for I began to read the historians almost as soon as I was able to read at all, and I found great pleasure and ease in verse. But as soon as I began to learn logic, I was greatly excited by the division and order in it. I immediately noticed, to the extent that a boy of 13 could, that there must be a great deal in it [5, p. 516].

*This project adapted from the one found in Barnett, J., Bezhanishvili, G., Leung, H., Lodder, J., Pengelley, D., Ranjan, D., "Historical Projects in Discrete Mathematics and Computer Science" in Resources for Teaching Discrete Mathematics, Hopkins, B. (editor), Mathematical Association of America, Washington, D.C., 2009.

His study of logic and intellectual quest for order continued throughout his life and became a basic principle to his method of inquiry. At the age of 20 he published *Dissertatio de arte combinatoria* (Dissertation on the Art of Combinatorics) in which he sought a *characteristica generalis* (general characteristic) or a *lingua generalis* (general language) that would serve as a universal symbolic language and reduce all debate to calculation. Leibniz maintained:

> If controversies were to arise, there would be no more need of disputation between two philosophers than between two accountants. For it would suffice to take their pencils in their hands, to sit down to their slates, and to say to each other: ... Let us calculate [14, p. 170].

The Leipzig-born scholar traveled extensively with diplomatic visits to Paris and London, and extended trips to Austria and Italy to research the history of the House of Brunswick. The years 1672–1676 were spent in Paris in an attempt to persuade King Louis XIV (1638–1715) not to invade Germany, but Egypt instead, although Leibniz was never granted an audience with the French king. During this time in Paris, however, the young German became acquainted with several of the leading philosophers of the day, acquired access to unpublished manuscripts of Blaise Pascal (1623–1662) and René Descartes (1596–1650), and met the renowned Christiaan Huygens (1629–1695), from whom he learned much about contemporary mathematics. During these years he laid the foundation of his calculus and the core of what would become his philosophical legacy.

Leibniz's invention of the differential and integral calculus is, in part, a product of his search for a universal language. Questions in the calculus can be reduced to the rules of calculation which the symbols for derivative, d, and integral, $\int$, satisfy. Sadly a priority dispute with Isaac Newton (1642–1727) over the invention of calculus cast a pall over Leibniz's later years. Moreover, he became a subject of ridicule with his philosophy that this is the best of all possible worlds bitterly satirized in Voltaire's (1694–1778) play *Candide*.

Let's turn to the universal genius's 1703 publication "Explication de l'arithmétique binaire, qui se sert des seuls caractères 0 et 1, avec des remarques sur son utilité, et sur ce qu'elle donne le sens des anciennes figures Chinoises de Fohy" [6, p. 223–227] (An Explanation of Binary Arithmetic Using only the Characters 0 and 1, with Remarks about its Utility and the Meaning it Gives to the Ancient Chinese Figures of Fuxi), which originally appeared in the journal *Memoires de l'Académie Royale des Sciences* [13]. Here again, with the reduction of arithmetic to expressions involving only zeroes and ones, we see a possible candidate for Leibniz's *characteristica generalis*. Of binary numeration, he writes "it permits new discoveries [in] ... arithmetic ... in geometry, because when the numbers are reduced to the simplest principles, like 0 and 1, a wonderful order appears everywhere." Concerning the binary calculations themselves " ... these operations are so easy that we shall never have to guess or apply trial and error, as we must do in ordinary division. Nor do we need to learn anything by rote." Certainly Leibniz was not the first to experiment with binary numbers or the general concept of a number base [7]. However, with base 2 numeration, Leibniz witnessed the confluence of several intellectual ideas of his world view, not just the *characteristica generalis*, but also theological and mystical ideas of order, harmony and creation [16]. Additionally his 1703 paper [13] contains a striking application of binary numeration to the ancient Chinese text of divination, the *Yijing* (*I-Ching* or *Book of Changes*).

Early in life Leibniz developed an interest in China, corresponded with Catholic missionaries there, and wrote on questions of theology concerning the Chinese. Surprisingly he believed that he had found an historical precedent for his binary arithmetic in the ancient Chinese lineations or 64 hexagrams of the *Yijing*. This, he thought, might be the origin of a universal symbolic language. A hexagram consists of six lines atop one another, each of which is either solid or broken, forming a total of 64 possibilities, while a grouping of only three such lines is called a trigram [cova]. Leibniz lists the eight possible trigrams in his exposition on binary arithmetic, juxtaposed with their binary equivalents.

He had been in possession of his ideas concerning binary arithmetic well before his 1703 publication. In 1679 Leibniz outlined plans for a binary digital calculating machine, and in 1697 he sent a congratulatory birthday letter to his patron Duke Rudolph August of Brunswick, in which he discusses binary numeration and the related creation theme with 0 denoting nothing and 1 denoting God [16]. Furthermore, Leibniz sent the French Jesuit Joachim Bouvet (1656–1730) an account of his binary system while Bouvet was working in China. The Jesuits are an educational order of Catholic priests, who, while in China, sought the conversion of the Chinese to Christianity, hopefully by the identification of an ancient theology common to both religions. Bouvet began a study of the *Yijing*, viewing this text as the possible missing link between the two religions [16]. It was from this Jesuit priest that Leibniz received the hexagrams attributed to Fuxi, the mythical first Emperor of China and legendary inventor of Chinese writing. In actuality, the hexagrams are derived from the philosopher Shao Yong's (1011–1077) *Huangji jingshi shu* (Book of Sublime Principle Which Governs All Things Within the World). Shortly after receiving Bouvet's letter containing the hexagrams and Bouvet's identification of a relation between them and binary numeration, Leibniz submitted for publication his 1703 paper "Explanation of Binary Arithmetic" [3, p. 44].

Exercise 1: Concerning the utility of the binary system, Leibniz cites an application to weighing masses. Suppose that a two-pan balance is used for weighing stones. A stone of unknown (integral) weight is placed on the left pan, while standard weights are placed only on the right pan until both sides balance. For example, if standard weights of 1, 4, 6 are used, then a stone of weight 7 on the left pan would balance the standard weights 1 and 6 on the right. Two standard weights with the same value cannot be used. Leibniz claims that all stones of integral weight between 1 and 15 inclusive can be weighed with just four standard weights. What are these four standard weights? Explain how each stone of weight between 1 and 15 inclusive can be weighed with the four standard weights. Make a table with one column for each of the four standard weights and another column for the stone of unknown weight. For each of the 15 stones, place an "X" in the columns for the standard weights used to weigh the stone.

Let's now read from an "Explanation of Binary Arithmetic," using a modified version of the Ching-Oxtoby translation [3, p. 81–86].

An Explanation of Binary Arithmetic Using only the Characters 0 and 1, with Remarks about its Utility and the Meaning it Gives to the Ancient Chinese Figures of Fuxi

By G.W. Leibniz

> Ordinary arithmetical calculations are performed according to a progression of tens. We use ten characters, which are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, that signify zero, one and the following numbers up to nine, inclusive. After reaching ten, we begin again, writing *ten* with 10, and ten times ten or *one hundred* with 100, and ten times one hundred or *one thousand* with 1000, and ten times one thousand with 10000, and so on.

Exercise 2: Write the numbers 1, 10, 100, 1000 and 10000 as powers of ten. Express your answer in complete sentences or with equations. What pattern do you notice in the exponents? (Question one appears just before the excerpt from Leibniz).

> But instead of the progression by tens, I have already used for several years the simplest of all progressions, that by twos, having found that this contributes to the perfection of the science of numbers. Thus I use no characters other than 0 and 1, and then, reaching two, I begin again. This is why *two* is written here as 10, and two times two or *four* as 100, and two times four or *eight* as 1000, and two times eight or *sixteen* as 1000, and so on.

Exercise 3: Write the numbers 1, 2, 4, 8 and 16 as powers of two. Express your answer in complete sentences or with equations. What pattern do you notice in the exponents? How do the exponents compare with those in question 2? How does the progression by twos compare with the standard weights in question 1?

> Here is the *Table of Numbers* according to this pattern, which we can continue as far as we wish.

Exercise 4: Compare the entries from 1 to 15 in Leibniz's "Table of Numbers"on the following page with the table for weighing stones that you constructed previously in question 1. Today a number written only with the characters 0 and 1 according to Leibniz's "progression of twos" is said to be written in binary notation, or base 2. Find the binary equivalents of the (base 10) numbers

$$34, \quad 64, \quad 100, \quad 1015\,.$$

Be sure to explain your work.

> At a glance we see the reason for the *famous property of the double geometric progression* in whole numbers, which states that given only one of these numbers in each degree, we can form all other whole numbers below the double of the highest degree. Since, as we would say, for example, 111 or 7 is the sum of four, two and one, and that 1101 or 13 is the sum of eight, four and one.

1	0	0	4
	1	0	2
		1	1
1	1	1	7

1	0	0	0	8
	1	0	0	4
			1	1
1	1	0	1	13

> This property is useful to investigators for weighing all kinds of masses with just a few weights or could be used in monetary systems to provide a range of change with just a few coins.

Table of Numbers

○	○	○	○	○	0	0
○	○	○	○	○	1	1
○	○	○	○	1	0	2
○	○	○	○	1	1	3
○	○	○	1	0	0	4
○	○	○	1	0	1	5
○	○	○	1	1	0	6
○	○	○	1	1	1	7
○	○	1	0	0	0	8
○	○	1	0	0	1	9
○	○	1	0	1	0	10
○	○	1	0	1	1	11
○	○	1	1	0	0	12
○	○	1	1	0	1	13
○	○	1	1	1	0	14
○	○	1	1	1	1	15
○	1	0	0	0	0	16
○	1	0	0	0	1	17
○	1	0	0	1	0	18
○	1	0	0	1	1	19
○	1	0	1	0	0	20
○	1	0	1	0	1	21
○	1	0	1	1	0	22
○	1	0	1	1	1	23
○	1	1	0	0	0	24
○	1	1	0	0	1	25
○	1	1	0	1	0	26
○	1	1	0	1	1	27
○	1	1	1	0	0	28
○	1	1	1	0	1	29
○	1	1	1	1	0	30
○	1	1	1	1	1	31
1	0	0	0	0	0	32

etc.

Exercise 5: Using modern notation, Leibniz's "double geometric progression" or "progression by twos" would be written 2^0, 2^1, 2^2, 2^3, $\ldots$, 2^n. Guess a simple formula for

$$2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^n$$

based on Leibniz's verbal description "that given only one of these numbers in each degree," we should be able to achieve all whole numbers "below the double of the highest degree." Prove that your guess is correct using only algebra (addition and multiplication). Hint: Multiply $(2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^n)$ by 1, where 1 is written as $2 - 1$.

Leibniz continues:

> This expression of numbers, once established, facilitates all kinds of operations.

For example, addition (1)

		1	1	0	6
		1	1	1	7
	.	.			
	1	1	0	1	13

		1	0	1	5
	1	0	1	1	11
.	.	.	.		
1	0	0	0	0	16

	1	1	1	0	14
1	0	0	0	1	17
1	1	1	1	1	31

For subtraction

1	1	0	1	13
	1	1	1	7
	1	1	0	6

1	0	0	0	0	16
	1	0	1	1	11
		1	0	1	5

1	1	1	1	1	31
1	0	0	0	1	17
	1	1	1	0	14

For multiplication (2)

		1	1	3
		1	1	3
		1	1	
	1	1		
	.			
1	0	0	1	9

	1	0	1	5
		1	1	3
	1	0	1	
1	0	1		
1	1	1	1	15

		1	0	1	5
		1	0	1	5
		1	0	1	
1	0	1	0		
	.				
1	1	0	0	1	25

For division

15	~~1~~	~~1~~	1	1		1	0	1	5
3	~~1~~	~~1~~	~~1~~	1					
		~~1~~	1						

All these operations are so easy that we shall never have to guess or apply trial and error, as we must do in ordinary division. Nor do we need to learn anything by rote here, as must be done in ordinary calculation, where, for example, it is necessary to know that 6 and 7 taken together makes 13, and that 5 multiplied by 3 gives 15, following the so-called Pythagorean table[1] that one times one is one. But here everything is found and proven from the source, just as we see in the preceding examples under the signs (1) and (2).

Exercise 6: Using your knowledge of base 10 addition, explain the examples of base 2 addition given above by Leibniz. What is the likely meaning of the dot Leibniz includes in certain columns for addition? Using binary arithmetic, compute $1101 + 1110$, without converting these numbers to base 10. Explain the examples for binary subtraction, multiplication and division above. Keep in mind that these should be base 2 analogues of base 10 procedures. Since Leibniz's example for division may be incomplete by today's standards, you may wish to supplement his work with additional steps, indicating clearly what multiples of 3 are subtracted from 15 in base 2. Finally, using binary arithmetic, compute the following.

$$11010 - 1101, \quad (1101) \cdot (11), \quad 1101 \div 101\,.$$

Be sure to explain your work. For the division problem, you may state what the remainder is in terms of a binary whole number, without writing it as a fraction.

[1]This is likely a reference to the multiplication table.

However, I am not at all recommending this manner of counting as a replacement for the ordinary practice of tens. For aside from the fact that we are accustomed to this, there is no need to learn what we have already memorized; the practice of tens is shorter, the numbers not as long. If we were accustomed to proceed by twelves or by sixteens, there would be even more of an advantage. As compensation for its length, however, calculation by twos, that is by 0 and by 1, is most basic for science; it permits new discoveries which become useful even in the practice of arithmetic, and especially in geometry, because when the numbers are reduced to the simplest principles, like 0 and 1, a wonderful order appears everywhere. For example, even in the Table of Numbers, we see in each column those periods which always reappear. In the first column it is 01, in the second 0011, in the third 00001111, in the fourth 0000000011111111, and so on. Small zeroes are put into the table to fill the void at the beginning of the column, and to mark these periods better. Lines are also traced in the table indicating that what these lines enclose always reoccurs below them. The square numbers, cubes and other powers, as well as the triangular numbers,[2] pyramidal numbers,[3] and other figurate numbers, also have similar periods, so that one can immediately write the tables without even calculating. A certain tedium at the beginning, which later serves to spare us calculation and to allow us to go by rule infinitely far, is extremely advantageous.

What is surprising in this calculation is that this arithmetic of 0 and 1 contains the mystery of lines of an ancient king and philosopher named Fuxi, who is believed to have lived more than four thousand years ago and whom the Chinese regard as the founder of their empire and of their sciences. There are several figures of lines that are attributed to him; they all go back to this arithmetic. But it is enough to place here the so-called figures of the eight Cova [trigrams], which are basic, and to add to these an explanation which is manifest, so that it is understood that a whole line — signifies unity or one, and that a broken line – – signifies zero or 0.

– –	—	– –	—	– –	—	– –	—
– –	– –	—	—	– –	– –	—	—
– –	– –	– –	– –	—	—	—	—
0	1	0	1	0	1	0	1
0	0	1	1	0	0	1	1
0	0	0	0	1	1	1	1
0	1	10	11	100	101	110	111
0	1	2	3	4	5	6	7

... [S]carcely two years ago I sent to the Reverend Father Bouvet, the famous French Jesuit living in Peking, my manner of counting by 0 and 1, and it was all he needed to recognize that this holds the key to Fuxi's[4] figures. So he wrote to me on November 14, 1701, sending me the great figure of this princely philosopher which goes to 64. ...

Although striking to Leibniz, the link between the 64 hexagrams of the *Yijing* and binary numeration appears today as only an intellectual curiosity. The base two system did not provide a common origin to Christianity and Confucianism, as Leibniz and Bouvet had sought.

The Electronic Age

John von Neumann (1903–1957) was a leading mathematician, physicist and engineer of the twentieth century, having contributed significantly to the foundations of quantum mechanics, the development of

[2]The sequence 1, 3, 6, 10, 15, ..., giving the number of dots in certain triangles [12, p. 49] forms the triangular numbers.
[3]The sequence 1, 4, 10, 20, 35, ..., giving the number of dots in certain pyramids [19, p. 76] forms the pyramidal numbers.
[4]The mythical first Emperor of China.

the atomic bomb, and the logical structure of the electronic digital computer [8, 10]. Born in Budapest Hungary, the young von Neumann showed a gift for mathematics, received a doctorate in the subject from the University of Budapest and a degree in chemical engineering from the *Eidgenössische Technische Hochschule* (Swiss Federal Polytechnic) in Zurich. He met the renowned David Hilbert (1862–1943) on a visit to Göttingen in 1926, after which he was offered the position of a *Privatdozent* (an un-salaried lecturer) at the University of Berlin and then at the University of Hamburg. In 1930 he visited the United States, accepting a salaried lectureship at Princeton University, a move which would shape the rest of his life.

Becoming a Professor of Mathematics at the prestigious Institute for Advanced Study (Princeton, New Jersey) in 1933, von Neumann was able to devote his time to the study of analysis, continuous geometry, fluid dynamics, wave propagation and differential equations. In 1943 he became a member of the Los Alamos Laboratory and helped develop the atomic bomb. The particular problem he faced, the implosion problem, was how to produce an extremely fast reaction in a small amount of the uranium isotope U^{235} in order to cause a great amount of energy to be released. In conjunction with Seth Neddermeyer, Edward Teller and James Tuck, this problem was solved with a high explosive lens designed to produce a spherical shock wave to cause the implosion necessary to detonate the bomb. Von Neumann's strength was his ability to model theoretical phenomena mathematically and solve the resulting equations numerically [8, p. 181], which required adroit skills in calculation.

Meanwhile in 1941 John William Mauchly (1907–1980), as a newly appointed Assistant Professor at the University of Pennsylvania's Moore School of Electrical Engineering, began discussions with graduate student John Presper Eckert (1919–1995) and others about the possibility of an electronic digital computing device that would be faster and more accurate than any existing machine, designed in part to meet the computing needs of the Ballistics Research Laboratory (BRL) of the Army Ordnance Department in Aberdeen, Maryland. With the help of mathematician and First Lieutenant Herman Goldstine (1913–), Mauchly's proposal for a high-speed vacuum-tube computer received funding from the BRL in 1943. The device was dubbed the Electronic Numerical Integrator and Computer (ENIAC). Tested in late 1945, and unveiled in 1946, the ENIAC was a behemoth containing 18,000 vacuum tubes and requiring 1,800 square feet of floor space for the computer alone [15, p. 133]. Arithmetic on the ENIAC was performed using the base 10 decimal system, requiring the ability to store ten different values for each digit of a numerical quantity. The multiplication table for all digits between zero and nine was also stored on the machine. The ENIAC was not programmable in the modern sense of a coded program, and contained no sub-unit similar to a present-day compiler. To alter its function, i.e., to implement a different numerical algorithm, external switches and cables had to be repositioned. Designs for a more robust machine may have been in place as the ENIAC went into production, but the rush to meet the needs of the war effort took precedence.

By serendipity, in the summer of 1944 Goldstine met von Neumann at a railway station in Aberdeen, both working on separate highly classified projects. Goldstine writes: "Prior to that time I had never met this great mathematician, but I knew much about him of course and had heard him lecture on several occasions" [8, p. 182]. After a discussion of the computing power of the ENIAC, von Neumann became keenly interested in this machine, and in late 1945 tested it on computations needed for the design of the hydrogen bomb. Von Neumann quickly became involved with the logical structure of the next generation of computing machinery, the Electronic Discrete Variable Automatic Computer (EDVAC), designed around the "stored program" concept. The instructions of an algorithm could be stored electronically on the EDVAC and then executed in sequential order. In this way, von Neumann had outlined a "universal computing machine" in the sense of Alan Turing (1912–1954), with the universal character referring to the machine's ability to execute any algorithmic procedure that could be reduced to simple logical steps. Turing first introduced a logical description of his universal computing machine in 1936 [17] as the solution to a problem posed by David Hilbert. Von Neumann, who had studied logic early in his career, was certainly aware of Turing's work, and in 1938 had offered Turing an assistantship at the Institute for Advanced Study [8, p. 271]. In 1945 von Neumann issued his white paper "First Draft of a Report on the EDVAC" [18] under the auspices of the University of Pennsylvania and the United States Army Ordnance Department. Although this draft was never revised, the ideas therein soon became known as von Neumann architecture in computer design. Let's read a few excerpts from this paper [18] related to binary arithmetic.

First Draft of a Report on the EDVAC

> 2.2 First: Since the device is primarily a computer, it will have to perform the elementary operations of arithmetic most frequently. There are addition, subtraction, multiplication and division: $+$, $-$, $\times$, $\div$. It is therefore reasonable that it should contain specialized organs for just these operations. At any rate a *central arithmetical* part of the device will probably have to exist, and this constitutes *the first specific part: CA*.
>
> 4.3 It is clear that a very high speed computing device should ideally have vacuum tube elements. Vacuum tube aggregates like counters and scalers have been used and found reliable at reaction times (synaptic delays) as short as a microsecond ($= 10^{-6}$ seconds).
>
> 5.1 Let us now consider certain functions of the first specific part: the central arithmetical part CA.
>
> The element in the sense of 4.3, the vacuum tube used as a current valve or *gate*, is an all-or-none device, or at least it approximates one: According to whether the grid bias is above or below cut-off; it will pass current or not. It is true that it needs definite potentials on all its electrodes in order to maintain either state, but there are combinations of vacuum tubes which have perfect equilibria: Several states in each of which the combination can exist indefinitely, without any outside support, while appropriate outside stimuli (electric pulses) will transfer it from one equilibrium into another. These are the so called *trigger circuits*, the basic one having two equilibria. The trigger circuits with more than two equilibria are disproportionately more involved.
>
> Thus, whether the tubes are used as gates or as triggers, the all-or-none, two equilibrium arrangements are the simplest ones. Since these tube arrangements are to handle numbers by means of their digits, it is natural to use a system of arithmetic in which the digits are also two valued. This suggests the use of the binary system.
>
> 5.2 A consistent use of the binary system is also likely to simplify the operations of multiplication and division considerably. Specifically it does away with the decimal multiplication table. In other words: Binary arithmetic has a simpler and more one-piece logical structure than any other, particularly than the decimal[5] one.

Exercise 7: Let a and b denote binary variables with one digit each. Using only the logical connectives $\wedge$ (and), $\vee$ (or) and $\sim$ (not), find a logical expression which gives the digit in the ones place (the right-hand digit) of $a + b$. Find a logical expression which gives the digit in the twos place (the left-hand digit) of $a + b$. Explain your answer.

[5] base 10

Extra Credit A: Find a pattern in the binary representation of the square numbers 1, 4, 9, 16, 25, … Leibniz claims to have found such patterns.

Extra Credit B: If in question 1 of the main project, standard weights can be placed on both sides of the balance, what four standard weights should be used in order to weigh all stones of integral weight between 1 and 40 inclusive?

Overview of LaTeX Required

This project will allow you to expand on the experiences of working with LaTeX in your first project. You may need a few commands outside of the normal ones we've discussed in class. A selection of binary operation symbols are provided below.

$\leq$	`\leq`	$\geq$	`\geq`	$\equiv$	`\equiv`	$\models$	`\models`
$\prec$	`\prec`	$\succ$	`\succ`	$\sim$	`\sim`	$\perp$	`\perp`
$\preceq$	`\preceq`	$\succeq$	`\succeq`	$\simeq$	`\simeq`	$\mid$	`\mid`
$\ll$	`\ll`	$\gg$	`\gg`	$\asymp$	`\asymp`	$\parallel$	`\parallel`
$\subset$	`\subset`	$\supset$	`\supset`	$\approx$	`\approx`	$\bowtie$	`\bowtie`
$\subseteq$	`\subseteq`	$\supseteq$	`\supseteq`	$\cong$	`\cong`	$\Join$	`\Join`
$\sqsubset$	`\sqsubset`[b]	$\sqsupset$	`\sqsupset`	$\neq$	`\neq`	$\smile$	`\smile`
$\sqsubseteq$	`\sqsubseteq`	$\sqsupseteq$	`\sqsupseteq`	$\doteq$	`\doteq`	$\frown$	`\frown`
$\in$	`\in`	$\ni$	`\ni`	$\propto$	`\propto`	$=$	`=`
$\vdash$	`\vdash`	$\dashv$	`\dashv`	$<$	`<`	$>$	`>`
$:$	`:`						

In addition to these symbols you will probably need to take a moment to read up on the formatting and creating of tables in LaTeX. A very nice guide on this subject can be found at `en.wikibooks.org/wiki/LaTeX/Tables`. I recommend using the `booktabs` package discussed at the bottom of the website. Another nice resource for working with LaTeX can be found at `http://www.artofproblemsolving.com/Wiki/index.php/LaTeX:About`. Don't forget that you can also reference the "Getting Started in LaTeX." video tutorial on the course blog. A copy of the TeX code for this document is available upon request. If you have any questions about the project or using LaTeX, feel free to comment in the associated LaTeX Writing Assignment blog post. (This way other people having the same questions can see your questions, and hopefully my, or other students', answers.)

References

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