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ENCOURAGING THE INTEGRATION OF COMPLEX NUMBERS IN UNDERGRADUATE ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. In this article we will demonstrate how introducing students to an alternative to the method of undetermined coefficients for solving for particular solutions of linear differential equations can be used to lead students to a more intuitive treatment of other topics in a typical undergraduate differential equations sequence. Given budget and staffing restrictions placed on smaller universities the techniques discussed offer a means for obtaining a higher level of student competency in complex numbers with a minimal resource and time investment.

1. Introduction

In many ways complex numbers and more specifically the properties of the complex exponential provide a means for condensing several techniques commonly encountered in an undergraduate differential equations course. Surprisingly, many of the contemporary texts for these courses relegate exercises involving complex numbers to end of chapter projects or deliver such exercises in relative obscurity compared to the abundance of real number examples they offer. While this may be fine for larger universities who can devote ample listings to courses in complex variables and Fourier analysis such luxuries are often not realized for two year colleges or smaller four year universities and the end result is students suffer from an incomplete mathematical foundation. In past years this may not have been a big issue, but with increasing pressure from physics and engineering applications complex operations and arithmetic are poised to assume a prominent role in the future of undergraduate mathematics. To this end, we will consider three areas in which a complex approach may be suitable.

2. Undetermined Coefficients

It is likely that even if you do not teach differential equations on a regular basis you are intimately familiar with the method of undetermined coefficients. In a first semester differential equations course the method arises as a means to solve for particular solutions of linear differential equations with constant coefficients of the type

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$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = f(x) \tag{2.1}$$

where $f(x)$ is normally restricted to linear combinations of the form

$$q(x)e^{kx}\cos(mx) \quad \text{or} \quad q(x)e^{kx}\sin(mx) \tag{2.2}$$

for some polynomial $q(x)$, with constants k and m (usually real). The method's importance cannot be understated given its applicability to the areas of mechanics and electrical circuits, but it can be terribly tedious and is generally unrevealing. For example, using the method of undetermined coefficients if one were to try and write down the form of a particular solution to the equation

$$y'' + 2y' + 2y = 5e^{-x}\sin(x) + 5x^3e^{-x}\cos(x) \tag{2.3}$$

they would quickly discover that there are no less than eight separate constants for which solutions must be obtained. Integrating complex arithmetic in this method is fairly straightforward as detailed in [5], but we will consider another method for accomplishing the same result in a more intuitive way. This result is expanded to a recursive algorithm in [3].

We will begin with a simple demonstration of the technique followed by a discussion of its importance relevant to this discussion. Consider the inhomogeneous second order linear differential equation

$$y'' - 3y' + 5y = e^{-3x}\sin 2x. \tag{2.4}$$

Finding a particular solution of an equation such as this is a fairly standard application of undetermined coefficients in most texts. We begin by rewriting the differential equation in its D-operator form and complexifying the right hand side. This reduces (2.4) to

$$p(D)y = e^{(-3+2i)x} \tag{2.5}$$

where $p(D) = D^2 - 3D + 5$ is the appropriate differential operator ($D = d/dx$). It can easily be shown that a particular solution, in this case called the exponential response, $\bar{y}$, for the complex equation (2.5) can be obtained using properties of the exponential and is given by

$$\bar{y} = \frac{e^{(-3+2i)x}}{p(-3+2i)}. \tag{2.6}$$

The next step is one that often gives students problems but is a necessary skill for students to possess, that is, recover the imaginary part of (2.6). Upon doing so one obtains the desired particular solution

$$\frac{18}{685}e^{-3x}\cos 2x + \frac{19}{685}e^{-3x}\sin 2x. \tag{2.7}$$

One advantage of this method is that by finding the real part of (2.6) one can obtain the exponential response for the same equation with $\cos 2x$ appearing on the right hand side. Finding the exponential response relies heavily on the application of exponentials and the properties of D. That is, it is straightforward to show that for $\alpha \in \mathbb{C}$,

$$D^k \left(e^{\alpha x} u(x)\right) = e^{\alpha x} (D + \alpha)^k u(x) \tag{2.8}$$

or more generally,

$$p(D) e^{\alpha x} u(x) = e^{\alpha x} p(D + \alpha) u(x). \tag{2.9}$$

This method can of course be expanded to solve equations of the form

$$p(D) y = e^{\alpha x} r(x) \tag{2.10}$$

for all the typical functions $r(x)$ to which the method of undetermined coefficients is desirable with the added feature of being easily generalizable to higher order equations. The benefits of emphasizing this technique and forcing students to come to terms with complex arithmetic are far reaching. This approach reinforces the concept of phase response in relation to amplifiers, filters, and resonance encountered later in the course not to mention provides a basis for motivating the concepts of transfer and weight functions associated with the Laplace transform.

3. Extension to the Laplace Transform

It is suggested that any discussion of the Laplace transform begin with inspection of the Fourier transform which is not possible unless a commitment to dealing with complex numbers has been established. The reason for this suggestion is that the Laplace transform is really just a special case of the Fourier transform. Consider an integrable function f defined on $\mathbb{R}$ with $f(t) = 0$ for $t < 0$. In this case, the Fourier transform of f is given by

$$\hat{f}(\omega) = \int_0^\infty f(t) e^{-i\omega t} \, dt, \tag{3.1}$$

where ω can be treated as complex, so that the integral (3.1) defines an analytic function on this domain. There are several forms of the Fourier transform which are suitable for discussion in this regard. In fact for many of the transform formulas there are both real and complex versions. The complex versions have a complex time domain signal and a complex frequency domain signal. The complex transforms are usually written using complex exponentials as shown in (3.1), though equivalent forms in terms of sine and cosine can be obtained as an application of Euler's relation. In the end, the specific version of the transform used in this paper is chosen for its easily observable connection to the Laplace transform formula used in [1], given by

$$\mathscr{L}\{f(t)\} = \int_0^\infty f(t) e^{-st} \, dt \tag{3.2}$$

defined for all values of s for which the improper integral converges. In the case of a complex ω, a restriction of ω to the half-plane Im $\omega < -\alpha$ means we can relax the integrability condition on f and (3.1) will converge for any functions $f(t)$

of exponential order. This is a statement commonly presented in textbooks with minimal support for its importance, but in this setting become apparent since a function $f(t)$ is of exponential order at $t \to +\infty$ if there exist constants a, b, and t_0 such that

$$|f(t)| \leq ae^{bt} \tag{3.3}$$

for $t \geq t_0$. In this situation it is convenient to make the change of variable $s = i\omega$ from which we obtain the Laplace transform

$$\mathscr{L}f(s) = F(s) = \hat{f}(-is) = \int_0^\infty f(t)e^{-st}\, dt. \tag{3.4}$$

It is worth mentioning that the Fourier inversion formula can be adapted to obtain the inverse Laplace transform as well. Once these ideas have been established, an elementary, but insightful treatment of the poles of the transform function $F(s)$ where s is complex is possible. We will explore this idea in an example.

The approach taken by most textbooks in obtaining $\mathscr{L}^{-1}F$, where $F(s)$ is a proper rational function of s (this is the most common example encountered), is to decompose $F(s)$ into partial fractions and use an inversion formula such as

$$\mathscr{L}^{-1}\ (s-a)^{-n-1} = e^{at}\frac{t^n}{n!}. \tag{3.5}$$

This method is neither enriching, nor revealing (much the same as undetermined coefficients) mathematically. As an aside it is worth mentioning the role of the exponential in "shifting" functions in the s and t domains. (Where have we seen something similar?) A more enriching approach would be to develop and use the fact that

$$\mathscr{L}^{-1}F(t) = \sum_{\text{all poles of F}} \text{Res}\ F(s)e^{st} \qquad (t > 0). \tag{3.6}$$

Put in perspective, (3.6) allows one to obtain the same information left to partial fractions applications in most texts without resorting to contour integration which would be rather sophisticated for students at this level. However, based on the examples above, i.e. $F(s)$ is a rational function of s, one can get the "residue/poles" treatment of the topic minus the contour integration, just by thinking of partial fractions–without actually implementing partial fractions. This idea is not particularly new or novel as it turns out. For a more thorough discussion of the "residue" idea as it relates to simplifying partial fractions computations the reader is directed to [4]. To clarify these points, consider the second order ordinary differential equation,

$$x'' + 2x' + 5x = f(t), \qquad x(0) = a, x'(0) = b. \tag{3.7}$$

Letting $F(s) = \mathscr{L}f(s)$ and $X(s) = \mathscr{L}x$ we have

$$s^2X(s) - (as + b) + 2sX(s) - 2a + 5X(s) = F(s) \tag{3.8}$$

which leads to

$$X(s) = \frac{F(s)}{s^2+2s+5} + \frac{as+b+2a}{s^2+2s+5}. \tag{3.9}$$

The zeros of s^2+2s+5 are $s = -1 \pm 2i$ and so by (3.6) we obtain

$$\begin{aligned} \mathscr{L}^{-1}\, \frac{1}{s^2+2s+5} &= \operatorname{Res}_{-1+2i}\, \frac{e^{st}}{s^2+2s+5} + \operatorname{Res}_{-1-2i}\, \frac{e^{st}}{s^2+2s+5} \\ &= \frac{e^{(-1+2i)t}}{4i} + \frac{e^{(-1-2i)t}}{-4i} \\ &= \frac{1}{2}e^{-t}\sin 2t. \end{aligned} \tag{3.10}$$

Also,

$$\mathscr{L}^{-1}\, \frac{as+b+2a}{s^2+2s+5} = ae^{-t}\cos 2t + \frac{1}{2}(a+b)e^{-t}\sin 2t. \tag{3.11}$$

which together with (3.10) gives the solution $x(t)$ when one considers the convolution of $f(t)$ with (3.9). For more information on the application of poles and residues to the Laplace transform the reader is referred to [2]. These methods are of clear value to students interested in engineering or physics as the flexibility and reliability of the Laplace transform makes it a powerful tool. Without the complex interpretation of the Laplace transform though the full potential of its value is not realized.

4. Beats

In studying the phenomenon of resonance in forced oscillations one will readily find a discussion of beats in most texts. A beat is an audible variation in the amplitude of a combined sound that occurs when the system is near resonance. This phenomenon is effectively the means by which musicians tune instruments. As far as I can tell there are no texts of the appropriate level which introduce this topic in the manner we will below.

A derivation of the beats phenomenon in most texts involves the use of the trigonometric identity

$$2\sin A\sin B = \cos(A-B) - \cos(A+B). \tag{4.1}$$

Consider a response of the form $x(t) = a\sin\omega_0 t + b\sin\omega_1 t$ where $\omega_1 \approx \omega_0$. This gives rise to a beat, but using the above method, we can only treat the case where $a = b$, i.e. the two oscillations have the same amplitude. (Technically, a trigonometric result can be obtained from writing the response in the form, $x(t) = a(\sin\omega_0 t + \sin\omega_1 t) + (b-a)\sin\omega_1 t$, but the derivation of a such a result is not nearly as intuitive as a complex treatment of the topic.) From a practical standpoint, the case $a = b$, while important, obscures much of the underlying richness of the model and subsequent applications involved. From the student's perspective as well, the manipulation of exponentials is more natural in practice than the use of trigonometric identities which are largely derived and understood from suitable complex identities in the first place.

To give a more complete treatment of the topic we will proceed using the complex exponential. Since we are studying two oscillations that are very near (in frequency) one another we will consider the sum of two complex exponentials of the form

$$x(t) = \alpha e^{i\omega_0 t} + \beta e^{i\omega_1 t} \tag{4.2}$$

where we think of $\omega_1 = \omega_0(1+\epsilon)$ for small ϵ. Here we assume $\alpha, \beta \in \mathbb{R}$ are constants. Under this assumption, we have

$$x(t) = e^{i\omega_0 t}\left(\alpha + \beta e^{i\epsilon\omega_0 t}\right) \tag{4.3}$$

which gives

$$|x(t)| = \sqrt{\alpha^2 + \beta^2 + 2\alpha\beta\cos(\epsilon\omega_0 t)} \tag{4.4}$$

since $|x| = x\bar{x}$ and $|e^{i\omega_0 t}| = 1$. This is the magnitude of the response often referred to as the envelope. Compare this with the equation for the envelope given in most texts. Now describing the beats phenomenon is much more straightforward. Observing (4.4), the maximum amplitude occurs for values of t which maximize the cosine term. That is, $t = 2n\pi/\epsilon\omega_0$ for $n = 0, 1, 2, \ldots$. The combined oscillation that occurs as a result has period modified by the value of ϵ, given by $2\pi/\epsilon\omega_0$. In this case we think of the beat as being a slight perturbation of the component oscillations. At the values of t listed above the maximum amplitude of the beat is $\alpha + \beta$, and when $t = (2n+1)\pi/\epsilon\omega_0$ we have a minimal amplitude, $|x(t)| = |\alpha - \beta|$. From this point any number of examples can be done and more clearly understood given the way in which the effect was derived. This also leads to a more interesting treatment of resonance as you can investigate what happens to the beat and its envelope as $\epsilon \to 0$.

5. Closing Remarks

The application of complex numbers and the complex exponential can unify many of the subjects taught in differential equations at the undergraduate level. While many of the topics described are standard topics in a junior level mechanics or electrical engineering course the degree to which these topics are treated as described in this paper are somewhat lessened in a conventional undergraduate differential equations course for mathematics majors. The immediate motivation for this approach stems from the rising need for general competency in complex operations at this level among members of the mathematical community. The ease with which these methods can be implemented is pedagogically relevant especially for smaller programs which can not afford a more complete treatment of complex variables at the undergraduate level. The methods described herein are but a small glimpse into the vast, untapped potential of experiencing differential equations from a complex perspective in undergraduate mathematics.

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