Extending the Set Partition Problem

An Introduction to the Beamer Class

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Objectives

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2. Introduce Bell numbers and the Bell triangle.
3. Extend the notion of partitions to multisets.
4. Asking more advanced questions–number of partitions with a specified largest part.
5. Extensions to research?

Set Partitions

Partition

A partition of a set S is a collection of non-empty, disjoint subsets of S whose union is S. More specifically, a set P of nonempty sets is a partition of X if

1. The union of the elements of P is equal to X, that is $\cup P = X$. (The elements of P are said to cover X.)
2. The intersection of any two distinct elements of P is empty, that is for $P_i, P_j \in P$, $P_i \cap P_j = \emptyset$ for all $i \neq j$. (We say the elements of P are pairwise disjoint.)

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n	$B(n)$
0	

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n	$B(n)$
0	1
1	1
2	

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n	$B(n)$
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n	$B(n)$
0	1
1	1
2	2
3	

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3	5

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4	

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4	15

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Bell Recursion

Bell Recursion Formula

The Bell numbers satisfy the recursion formula

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k)$$

where $B(0) = 1$ and $B(1) = 1$.

So to answer our previous question, in the case of $B(5)$, we have

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$$B(5) = \binom{4}{0}B(0) + \binom{4}{1}B(1) + \binom{4}{2}B(2) + \binom{4}{3}B(3) + \binom{4}{4}B(4)$$

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Bell Triangle

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How does this approach change if we allow elements of our set to be repeated?

Multiset

Multiset

A multiset, or mset for short, is an unordered collection of elements in which, unlike a standard (Cantorian) set, elements are allowed to repeat. In other words, an mset is a set to which elements may belong more than once. For example, an mset containing one occurrence of a, two occurrences of b, and three occurrences of c is notationally written as

$$[[a, b, b, c, c, c]] \quad \text{or} \quad [a, b, c]_{1,2,3} \quad \text{or} \quad [a^1, b^2, c^3] \quad \text{or} \quad [a1, b2, c3],$$

depending on one's taste and convenience.

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But in general, not much is known about permutations of multisets and a fleshing out of combinatorical results involving msets is still fairly new.

Partitions of Multisets

How does the idea of a set partition generalize to multisets?

Surprisingly, this question is related to computing all possible factorizations of a given integer.

Relating Factorizations to Multiset Partitions

Consider representing integers by multisets of their prime factors. For example, $30 = \{2, 3, 5\}$, and $12 = \{2, 2, 3\}$. We must use multisets of prime factors rather than sets, since we want to be able to distinguish between, say, $6 = \{2, 3\}$ and $12 = \{2, 2, 3\}$. Given an integer n and its multiset of prime factors, a factorization of n corresponds to a particular grouping of the prime factors into subsets. For example, for $n = 30$, the factorization $10 \cdot 3$ corresponds to the grouping $\{\{2, 5\}, \{3\}\}$.

Multiset Partition Example

Example

As an example, we can list all the partitions of $\{2, 2, 3\}$,

$$\{\{\{2, 2, 3\}\}, \{\{2, 2\}, \{3\}\}, \{\{2\}, \{2, 3\}\}, \{\{2\}, \{2\}, \{3\}\}\}.$$

These correspond to the factorizations $12 = 4 \cdot 3 = 2 \cdot 6 = 2 \cdot 2 \cdot 3$, and in general its not hard to see that factorizations of an integer exactly correspond to partitions of its multiset of prime factors.

Notice, there are only four partitioning sets whereas for a normal set $B(3) = 5$. So at this point, it becomes obvious that the Bell recursion formula no longer applies.

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- What about recursive formulas to determine the number of set partitions with largest part k? Once again, it has been done.
- What about recursive formulas to determine the number of multiset partitions with largest part k? Score! To the best of my knowledge this is unknown (or more likely just undone since this material is fairly new)

Further Resources

- Beamer Class Homepage:
  `http://latex-beamer.sourceforge.net`
- Beamer Quick Start Page:
  `http://www.math.umbc.edu/~rouben/beamer`
- Wolfram Demonstrations Page:
  `http://demonstrations.wolfram.com`
- Wolfram Computational Site:
  `http://www.wolframalpha.com`