The Study of the Differential Equation for the Spiral Flight of Insects

David Suazo, Ayrea Towell, and Kristin Holz

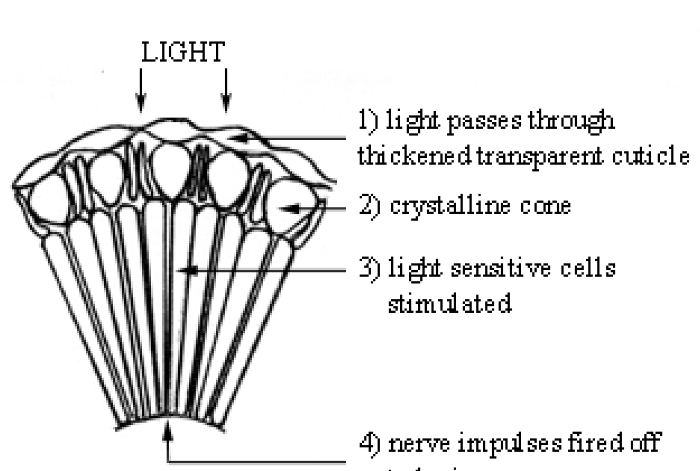
Department of Mathematics, Abilene Christian University

The Big Question

How do insects fly? While you might normally think of insects traveling in a straight line towards an object, insects actually travel in a spiral flight pattern near light sources. Why is it that insects fly in such a pattern?

Insects and Their Eyes

Ommatidia: An insect's eye is made up of over 20,000 units called ommatidium. Each of these units has a lens that focuses light rays into the rhabdom. An opaque wall around the rhabdom blocks out all light rays that would enter from the sides. In order to see, an insect must have light rays shining parallel into any of the ommatidia. Sight like this allows an insect to detect fast movement to a more precise level than that of the human eye. Rapid motion vision enables an insect to see accurately while flying. This sight requires that some light ray is perpendicular to at least one ommatidium, and thus when an insect is flying, at least one ommatidium is perpendicular to a light ray. The figures below show a close up view of an insect's eye and a detailed diagram of the structure of the ommatidia.

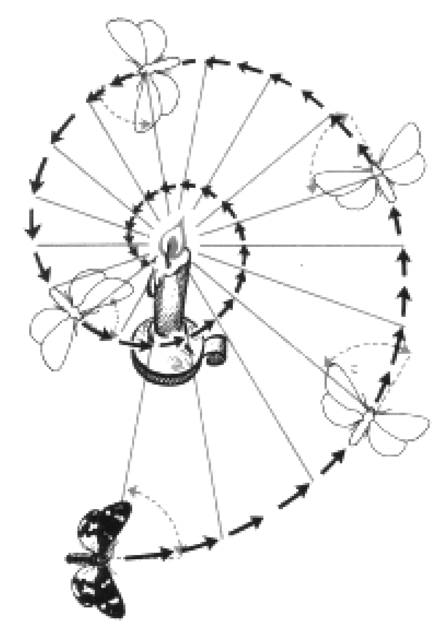


Light travelling down ommatidia

Assumptions of the Model

For this problem we make the following assumptions:
- The light source is at the center of the coordinate system.
- The insect is small and its velocity is relatively slow.

2D Model



Our insect has radius vector $\bar{x} = (x, y)$ and flies by the law

$$x(t) = r(t)\cos(\theta(t)), y(t) = r(t)\sin(\theta(t))$$

where t is time and r, θ are polar coordinates. Now $\bar{x}'$ is the tangent vector that defines the direction of flight. We need an angle α that is between $\bar{x}'$ and $-\bar{x}'$ to stay constant for the flight to be circular. Through this we have

$$x' = r'\cos(\theta) - r\sin(\theta)\theta', y' = r'\sin(\theta) + r\cos(\theta)\theta'. \quad (1)$$

Using the dot product to find α we can show the following

$$\frac{\bar{x} \cdot \bar{x}'}{||\bar{x}||||\bar{x}'||} = -\cos(\alpha)$$

It can be shown that the right hand side of this equation gives us our main differential equation for the flight of the insect with an equiangular motion which is the following,

$$\frac{r'}{\sqrt{(r')^2 + (\theta')^2 r^2}} = -\cos(\alpha). \quad (2)$$

Using this equation and assuming that the insect flies in a constant speed we can find that

$$r' = -v\cos(\alpha).$$

Therefore,

$$r(t) = -(v\cos 9\alpha)t + r_0 \quad (3)$$

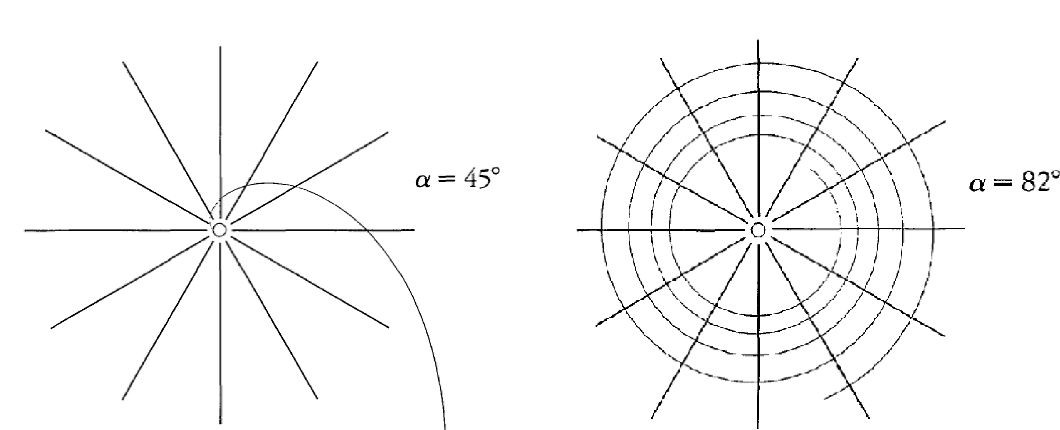
where $r_0 = r(0)$. Plugging this equation into equation (1) we find $\theta(t)$.

$$\theta(t) = \theta_0 - \ln(1 - \frac{vt}{r_0}\cos(\alpha))\tan(\alpha) \quad (4)$$

Now the trajectory of the flight is of the form $r = r(\theta)$ and to find this equation we eliminate time from equations (3) and (4) to get the equation that shows the trajectory is an equiangular spiral.

$$r(\theta) = r_0 e^{\cot\alpha(\theta_0 - \theta)}. \quad (5)$$

The following figures demonstrate the two-dimensional trajectory of the insect along two different values of α.

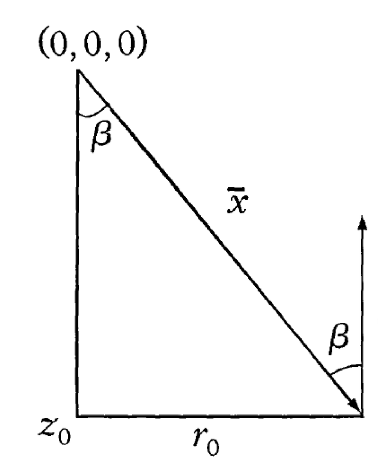


3D Model

Position Trajectory: When the insect's trajectory is modeled in three dimensions, its radius vector becomes $x = (x, y, z)$ or $(rcos\theta, rsin\theta, z)$. Differentiating this vector with respect to time, the velocity vector for the insect's motion becomes $\bar{x}' = (r'cos(\theta) - rsin(\theta)\theta', r'sin(\theta) + rcos(\theta)\theta', z')$. This allows us to take the dot product of the radius vector and velocity vector in order to relate them to α.

$$\frac{\bar{x} \cdot \bar{x}'}{\|x\|\|\bar{x}'\|} = \frac{rr' + zz'}{\sqrt{r^2 + z^2}\sqrt{(r')^2 + (\theta')^2 r^2 + (z')^2}} = -cos\alpha \quad (6)$$

In addition to keeping α constant, the insect keeps a constant angle β between the direction to the light source and the vertical axis as seen in the figure below where the light is at the origin and k represents the unit vector of the z axis.

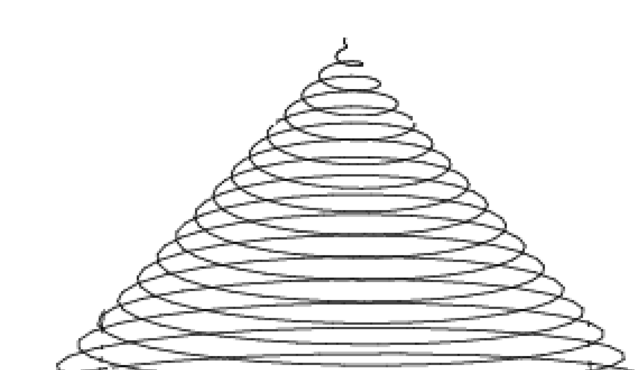


Using the dot product, the vertical axis vector $\bar{k} = (0, 0, 1)$ and position vector $x = (rcos\theta, rsin\theta, z)$ can be compared to the angle β. From this relation, we can determine that the position trajectory lies on a right circular cone with its vertex at the origin and its base opening below the vertex. To find the specific path along this cone, each dot product can be manipulated using trigonometry and algebra to find the equations of the trajectory,

$$x = r_0 e^{-m\theta}\cos\theta, \qquad y = r_0 e^{-m\theta}\sin\theta, \qquad z = z_0 e^{-m\theta} \quad (7)$$

where $r_0 = -\tan\beta z_0$ and $m = \sin\beta\cot\alpha$.

These equation result in the trajectory shown in the figure below.



Motion Trajectory: Incorporating a constant velocity $\|\bar{x}'\|$ we can derive the trajectory equations as functions of time. Here $a = v\cos\alpha\cos\beta$, $b = v\cos\alpha\sin\beta$, and $c = \frac{\tan\alpha}{\sin\beta}$.

$$x = (r_0 - bt)cos[\theta_0 - cln(1 + at/z_0)] \quad (8)$$
$$y = (r_0 - bt)sin[\theta_0 - cln(1 + at/z_0)] \quad (9)$$
$$z = z_0 + at \quad (10)$$

Conclusion

Insects fly in a spiral path towards a light source. This flight pattern can be modeled with a system of differential equations. While the system is not completely accurate, the insect will eventually fly into the light source whereas the mathematical model will result in an orbit around the light source, this model can be used as an illustration of flight for small insects with relatively slow velocities. Another inaccuracy in this model of insect flight occurs when larger and faster insects are used due to the added parameters.

At a certain constant angle of the insect's flight, the spiral pattern formed results in the Fibonacci series. This is an interesting feat of nature which continues to show up in numerous unexpected places, such as the seed pattern on a sunflower as pictured below.

Text Reference

This project has been adapted in part from the resource below:

[1] Khristo N. Boyadzhiev, Spirals and Conchospirals in the Flight of Insects, *The College Mathematical Journal*, Vol. 30, No. 1, 1999.

Picture References

The pictures and figures have been used and adapted from the following sources listed in order of appearance:

[1] `http://theshadoxhurstgarden.blogspot.com/2010/10/migrant-hawker-compound-eye.html`

[2] `http://boston.com/bigpicture/2008/11/peering_into_the_micro_world.html`

[3] `http://library.thinkquest.org/28030/eyeevo.htm`

[4] `http://www.entomologia.org/bugs_in_the_news/`

[5] Khristo N. Boyadzhiev, Spirals and Conchospirals in the Flight of Insects, *The College Mathematical Journal*, Vol. 30, No. 1, 1999.

[6] `http://mathworld.wolfram.com/ConicalSpiral.html`

[7] `http://itherin.wordpress.com/page/2/`