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Positive Solutions of a Left Focal Second Order Boundary Value Problem

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Abstract

Using appropriately defined functionals, a cone expansion and compression fixed point theorem is applied to obtain a positive solution and bounds on the solution of a left focal second order boundary value problem. We compare our results to those known for a similar right focal boundary value problem.

Key words: boundary value problem, focal, second order, differential equation, cone expansion and compression.
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Many of the problems in nonlinear analysis arise from the inspection of equations or problems that somehow model real phenomena. Many problems including but not limited to, fluid dynamics, gas diffusion through a porous media, thermal self ignition of a chemically active mixture of gases, catalytic theory, and diffusion of heat are positive-dependent; that is, only positive solutions are of any significance.

In this paper we consider the existence of positive solutions for a second order differential equation with left focal boundary conditions. Our main result yields the existence of a positive solution as well as bounds for the solution. Our techniques make use of the fixed point theorem of cone expansion and compression of functional type found in [3]. We apply the theorem to a completely continuous operator whose fixed points are solutions for our problem. The bounds for our solutions are a direct result of the extension of this theorem. (see [4])

Obtaining fixed points for the operators considered in this paper is a topic that has been extensively researched. In [14], Krasnosel'skii considered a similar integral operator A which was bounded above and below by order preserving operators A_1 and A_2. If A_1 and A_2 could be shown to have alternating regions of slow and then rapid growth it was possible to recover disjoint order intervals left invariant by the operator A. Via the Schauder fixed point theorem, Krasnosel'skii showed that the operator A has a fixed point in each of the intervals which A left invariant. This result led numerous mathematicians to investigate under what conditions operators of this type would have not only one fixed point, but multiple fixed points. We refer the reader to [2], [9], [10], and [17] for representatives of papers in this area.

In the literature it is not uncommon to refer to the Krasnosel'skii fixed point theorem as the fixed point theorem of cone expansion and compression of norm type. Recently several authors [3], [4], [5] have generalized this theorem by replacing the norm conditions with suitably defined functionals. According to [3], these functionals allow a greater flexibility in applying the Krasnosel'skii result especially in applications to boundary value problems. It is these applications in which we are interested.

1 Preliminaries

We begin with some preliminary definitions and background results on cones and completely continuous operators. We also state the fixed point theorem of cone expansion and compression of functional type.

Definition 1.1 *Let E be a real Banach space. A nonempty, closed, convex set $P \subset E$ is called a cone if it satisfies the following:*

(i) *If $x \in P$, then $\lambda x \in P$ for all $\lambda \geq 0$.*

(ii) *If $x \in P, -x \in P$ then $x = 0$.*

Definition 1.2 *A cone P of a real Banach space E is said to be normal if there exists $\delta \geq 0$ such that $\|x + y\| \geq \delta$ for all $x, y \in P$ with $\|x\| = \|y\| = 1$.*

It is well known that for $P \subset E$ a normal cone in a real Banach space E, P is normal if and only if the norm of the Banach space E is semimonotone. That is, there exists a constant $N \geq 0$ such that $0 \leq x \leq y$ implies that

$$\|x\| \leq N\|y\|.$$

A proof of this result can be found in [7].

Definition 1.3 *Let E be a real Banach space. An operator $A : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into precompact sets. Additionally, the operator A is called increasing on a domain D if for $P \subset E$, $x_1, x_2 \in D \subset P$, then $A : D \to E$ has the property that for $x_1 \leq x_2$, $Ax_1 \leq Ax_2$.*

Definition 1.4 *A non-negative continuous functional α defined on a cone P in a Banach space E is a map*

$$\alpha\colon P \to [0, \infty)$$

that is continuous. Additionally, the functional is said to be concave if it satisfies

$$\alpha(\lambda x + (1-\lambda)y) \geq \lambda\alpha(x) + (1-\lambda)\alpha(y), \quad 0 \leq \lambda \leq 1.$$

The existence of completely continuous increasing operators which are invariant under the action of non-negative continuous functionals in regions of a cone are the basis for our results.

2 Fixed Point Theorem of Cone Expansion and Compression of Functional Type

In this section we will state the fixed point theorem of cone expansion and compression of functional type due Anderson, Avery, and Krueger in [3] and [4]. A majority of the applications presented in this paper are consequences of this theorem. A proof of this result can be found in [3] and [4]. In the theorem that follows as an analog to order intervals, the authors consider sets of the form

$$P(\gamma, \alpha, r, R) = \{x \in P\colon r < \alpha(x) \text{ and } \gamma(x) < R\}$$

where α and γ are non-negative continuous functionals (often the maximum and minimum of a function over a specific interval) and $r, R \in \mathbb{R}^+$.

Theorem 2.1 *Let P be a normal cone in a real Banach space E, and let α and γ be nonnegative continuous functionals on P. Assume $P(\gamma, \alpha, r, R)$ as above is a nonempty bounded subset of P,*

$$A : \overline{P(\gamma, \alpha, r, R)} \to P$$

is a completely continuous operator and

$$\overline{P(\alpha, r)} \subseteq P(\gamma, R).$$

If one of the two conditions,

$(H1)$ $\alpha(Ax) \leq r$ *for all* $x \in \partial P(\alpha, r)$, $\gamma(Ax) \geq R$ *for all* $x \in \partial P(\gamma, R)$,

$$\inf_{x \in \partial P(\gamma, R)} \|Ax\| > 0,$$

and for all $y \in \partial P(\alpha, r)$, $z \in \partial P(\gamma, R)$, $\lambda \geq 1$, and $\mu \in (0, 1]$, the functionals satisfy the properties

$$\alpha(\lambda y) \geq \lambda \alpha(y), \;\; \gamma(\mu z) \leq \mu \gamma(z), \;\; \textit{and} \;\; \alpha(0) = 0,$$

$(H2)$ $\alpha(Ax) \geq r$ *for all* $x \in \partial P(\alpha, r)$, $\gamma(Ax) \leq R$ *for all* $x \in \partial P(\gamma, R)$,

$$\inf_{x \in \partial P(\alpha, r)} \|Ax\| > 0,$$

and for all $y \in \partial P(\alpha, r)$, $z \in \partial P(\gamma, R)$, $\lambda \in (0, 1]$ and $\mu \geq 1$, the functionals satisfy the properties

$$\alpha(\lambda y) \leq \lambda \alpha(y), \;\; \gamma(\mu z) \geq \mu \gamma(z), \;\; \textit{and} \;\; \gamma(0) = 0,$$

is satisfied, then A has at least one positive fixed point x^* such that

$$r \leq \alpha(x^*) \;\; \textit{and} \;\; \gamma(x^*) \leq R.$$

Moreover, suppose there exist $x_l, x_u \in P$ such that $\overline{P(\gamma, \alpha, r, R)} \subseteq [x_l, x_u]$.

$(E1)$ *If there exists an increasing, completely continuous operator* $U : [x_l, x_u] \to P$ *such that* $Ax \le Ux$ *for all* $x \in [x_l, x_u]$ *and* $U^2 x_u \le U x_u$*, then*

$$x^* \le x_u^* \le U^n x_u,$$

where $n \in \mathbb{N}$ *and* $x_u^* = \lim_{n\to\infty} U^n x_u$.

$(E2)$ *If there exists an increasing, completely continuous operator* $L : [x_l, x_u] \to P$ *such that* $Lx \le Ax$ *for all* $x \in [x_l, x_u]$ *and* $Lx_l \le L^2 x_l$*, then*

$$L^n x_l \le \ x_l^* \le x^*,$$

where $n \in \mathbb{N}$ *and* $x_l^* = \lim_{n\to\infty} L^n x_l$.

3 Second Order Existence Results

Consider the second order boundary value problem,

$$y''(t) + f(y(t)) = 0, \quad t \in [0,1], \tag{1}$$

$$y(1) = y'(0) = 0, \tag{2}$$

where $f : \mathbb{R} \to [0, \infty)$ is continuous. By a positive solution of (1), (2) we mean, $y \in C^2[0,1]$, y satisfies (1) on $[0,1]$ and the boundary conditions (2), y is non-negative on $[0,1]$ and positive on some subinterval of $[0,1]$. The solutions of (1), (2) are the fixed points of the operator A given by

$$Ay(t) := \int_0^1 G(t,s) f(y(s)) ds, \quad t \in [0,1],$$

where, $G(t,s)$ is the Green's function for $-y'' = 0$ satisfying the boundary conditions (2). In particular, the Green's function is given by

$$G(t,s) = \begin{cases} 1-s, & t \le s, \\ 1-t, & s \le t. \end{cases}$$

Let $E = C[0,1]$ with supremum norm and define the cone $P \subset E$ by

$$P := \{y \in E \ : \ \text{y is concave, nonnegative, and non-increasing}\}.$$

As motivation for the defining of our functionals α, γ we remark that for $y \in P$,

$$y(t) \geq (1-t)\|y\|, \quad 0 \leq t \leq 1.$$

Let $\eta \in (0,1)$ and define $\alpha, \gamma : P \to \mathbb{R}$ by

$$\alpha(y) := \min_{t \in [0,1-\eta]} y(t) = y(1-\eta)$$

and

$$\gamma(y) := \max_{t \in [0,1]} y(t) = \|y\| = y(0).$$

We now provide the main result of the paper.

Theorem 3.1 *Suppose there exist positive real numbers r, R with $r \leq \eta R$, and a continuous $f : \mathbb{R} \to [0,\infty)$, such that the following conditions are met:*

(*i*) $f(w) \leq 2R$ *for* $w \in [0, R]$,

(*ii*) $f(w) \geq \frac{r}{\eta(1-\eta)}$ *for* $w \in [r, R]$.

Then the left focal boundary value problem (1), (2) *has at least one positive solution y^* such that*

$$\frac{r}{\eta(1-\eta)} \left(\frac{1-(\eta^2+t^2)}{2} \right) \leq y^*(t) \leq R(1-t^2), \quad t \in [0, 1-\eta]$$

and

$$\frac{r(1-t)}{\eta} \leq y^*(t) \leq R(1-t^2), \quad t \in [1-\eta, 1].$$

Proof: Let $y \in P$. We claim that $A : P \to P$. Let $w(t) = Ay(t)$. Then $w(1) = w'(0) = 0$ and $w''(t) = -f(y(t)) \leq 0$. Hence $w'(t)$ is decreasing and since $w'(t) \leq 0$ for all $t \in [0,1]$, we have $w(t)$ decreasing and $w(t) \geq 0$ by the boundary conditions. Hence $w(t) \in P$. Also, the cone P is normal since the norm of the Banach space E is semi-monotone. We verify a series of claims for the result.

Claim 1: $\overline{P(\alpha, r)} \subset P(\gamma, R)$.

Recall, that for all $y \in P$, $y(t) \geq (1-t)\|y\|$. Let $y \in \overline{P(\alpha, r)}$. We have,

$$\begin{aligned} r \geq \alpha(y) = y(1-\eta) &\geq (1-(1-\eta))\|y\| \\ &= \eta\|y\| \\ &= \eta y(0) \\ &= \eta\gamma(y). \end{aligned}$$

Thus $R \geq \frac{r}{\eta} \geq \gamma(y)$. So $y \in P(\gamma, R)$. We simply note at this point that clearly

$$\alpha(\lambda y) = \lambda(\alpha(y)), \gamma(\mu z) = \mu\gamma(z), \gamma(0) = 0$$

for all $y \in P(\alpha, r), z \in \partial P(\gamma, R), \lambda \in (0, 1]$ and $\mu \geq 1$. We now propose another claim.

Claim 2: For $y \in \partial P(\alpha, r)$ (i.e. $\alpha(y) = y(1-\eta) = r$) we have $\alpha(Ay) \geq r$.

Let $y \in \partial P(\alpha, r)$. Then

$$\begin{aligned} \alpha(Ay) = Ay(1-\eta) &= \int_0^1 G(1-\eta, s) f(y(s)) ds \\ &\geq \int_0^{1-\eta} G(1-\eta, s) f(y(s)) ds \\ &\geq \frac{r}{\eta(1-\eta)} \int_0^{1-\eta} G(1-\eta, s) ds \\ &= \frac{r}{\eta(1-\eta)} \eta(1-\eta) \\ &= r. \end{aligned}$$

The above arguments also yield

$$\inf_{y \in \partial P(\alpha, r)} \|Ay\| \geq r \geq 0.$$

Claim 3: For $y \in \partial P(\gamma, R)$ (i.e. $\gamma(y) = y(0) = R$) we have $\gamma(Ay) \leq R$.

Let $y \in \partial P(\gamma, R)$. Then

$$\begin{aligned}\gamma(Ay) = Ay(0) &= \int_0^1 G(0,s)f(y(s))ds \\ &\leq 2R\int_0^1 G(0,s)ds \\ &= 2R\frac{1}{2} \\ &= R.\end{aligned}$$

By Theorem 2.1 we are guaranteed the existence of a positive solution $y^*(t)$ to (1), (2). Now define the increasing completely continuous operators $L : P \to P$, $U : P \to P$ by

$$Ly(t) := \int_0^{1-\eta} G(t,s)\frac{r}{\eta(1-\eta)}ds$$

and

$$Uy(t) := \int_0^1 G(t,s)2Rds$$

Then for all $y \in P(\gamma, \alpha, r, R)$

$$Ly \leq Ay \leq Uy.$$

Moreover, if we define $y_u, y_l \in P$ by

$$y_u(s) := R$$

and

$$y_l(s) := \begin{cases} r, & s \leq 1-\eta, \\ \frac{r(1-s)}{\eta}, & 1-\eta \leq s, \end{cases}$$

then as a consequence of the concavity of P, we have $P(\gamma, \alpha, r, R) \subset [y_l, y_u]$. Clearly, by the the above claims we have $Uy_u \leq y_u$ and since $Ly_l(0) \geq r$ and $Ly_l(1-\eta) \geq r$ then $Ly_l \geq y_l$. This gives

$$Ly_l(t) = \frac{r}{\eta(1-\eta)}\int_0^{1-\eta} G(t,s)ds \leq y^*(t) \leq 2R\int_0^1 G(t,s)ds = Uy_u(t)$$

as desired. Note that to obtain the bounds used in Theorem 3.1 we use the properties of the Green's function below.

$$\int_0^1 G(t,s)ds = \frac{1-t^2}{2}$$

and

$$\int_0^{1-\eta} G(t,s)ds = \begin{cases} \frac{1-(\eta^2+t^2)}{2}, & t \in [0, 1-\eta], \\ (1-t)(1-\eta), & t \in [1-\eta, 1]. \end{cases}$$

Remark: We want to take this opportunity to comment on the purpose and role of the condition that $r \leq \eta R$, and for what values of η does this relationship make sense given that

$$\frac{r}{\eta(1-\eta)} \leq f(w) \leq 2R.$$

We note that

$$\frac{1}{\eta(1-\eta)} \geq 4$$

and so we have

$$4r \leq \frac{r}{\eta(1-\eta)} \leq 2R.$$

Given the above we have $r \leq \frac{R}{2}$ and so for values of $\eta \in [\frac{1}{2}, 1)$ our theorem makes sense and gives nice bounds for the non-linearity f over the intervals $[0, R]$ and $[r, R]$. We note that for $\eta = \frac{1}{2}$, the lower and upper bounds on f are the same. We have assumed this throughout the paper.

A similar result for the right focal second order boundary value problem,

$$y''(t) + f(y(t)) = 0, \quad t \in [0, 1], \tag{3}$$

$$y(0) = y'(1) = 0, \tag{4}$$

where $f : \mathbb{R} \to [0, \infty)$ is continuous, is considered in [4] and lends itself nicely to the approach used in this paper. We state the results of [4] in similar fashion to Theorem 3.1.

Theorem 3.2 *Suppose there exist positive real numbers r, R with $r \leq \eta R$, and $f : \mathbb{R} \to [0, \infty)$, such that the following conditions are met:*

(i) $f(w) \leq 2R$ *for* $w \in [0, R]$,

(ii) $f(w) \geq \frac{r}{\eta(1-\eta)}$ *for* $w \in [r, R]$.

Then the right focal boundary value problem (3), (4) *has at least one positive solution* y^* *such that*

$$\frac{rt}{\eta} \leq y^*(t) \leq Rt(2-t), \quad t \in [0, \eta]$$

and

$$\frac{r}{\eta(1-\eta)}\left(t - \frac{(\eta^2 + t^2)}{2}\right) \leq y^*(t) \leq Rt(2-t), \quad t \in [\eta, 1].$$

Remark: We observe that for both the left focal problem (1), (2) and the right focal problem (3), (4) the bounds on the non-linearity $f(w)$ are identical. This is not without coincidence as the upper bound is fixed and the lower bound is symmetric about $1 - \eta$. A suitable cone, K, for the right focal boundary value problem (3), (4) is as you would expect,

$$K := \{y \in E \ : \ \text{y is concave, nonnegative, and non-decreasing}\}.$$

The concavity of the cones P and K for both problems is the motivation for our techniques. These examples illustrate well the nature of second order focal boundary value problems and exhibit many of the characteristics of optics problems in physics. In fact, the focal boundary value problems (1), (2) and (3), (4) are intimately related to the first and second focal points of a converging lens.

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