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Group #2

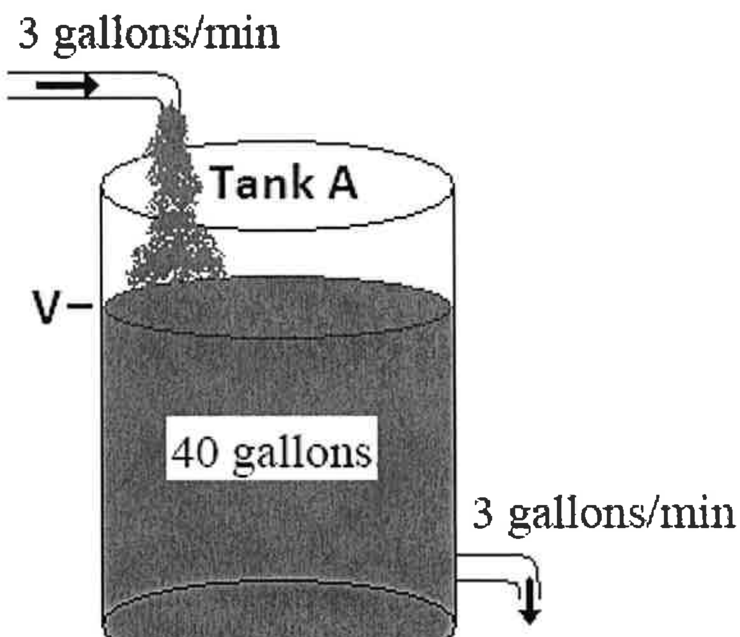
with(DEtools) :
with(plots) :

Section 1: Evolution of the Mixing Problem

First Order Mixing Problem

In solving a mixing problem using differential equations, beginning with the first-order equation was a simple way to model a single tank with one point of water entry and one point of solution exit. The model consists of:

Consider a single tank which has a constant volume of 40 gallons. The tank initially has 2 lbs. of salt in it. Pure water is running into it at a rate of 3 gallons/min and salt solution drains out at a rate of 3 gal/min. Assume perfect mixing and solve for x(t), the salt content in the tank.



$x'(t) =-\frac{3}{40}\cdot x(t)$

$$D(x)(t) = -\frac{3}{40}\,x(t) \qquad \textbf{(1.1.1)}$$

dsolve(**(1.1.1)**, $x(t)$)

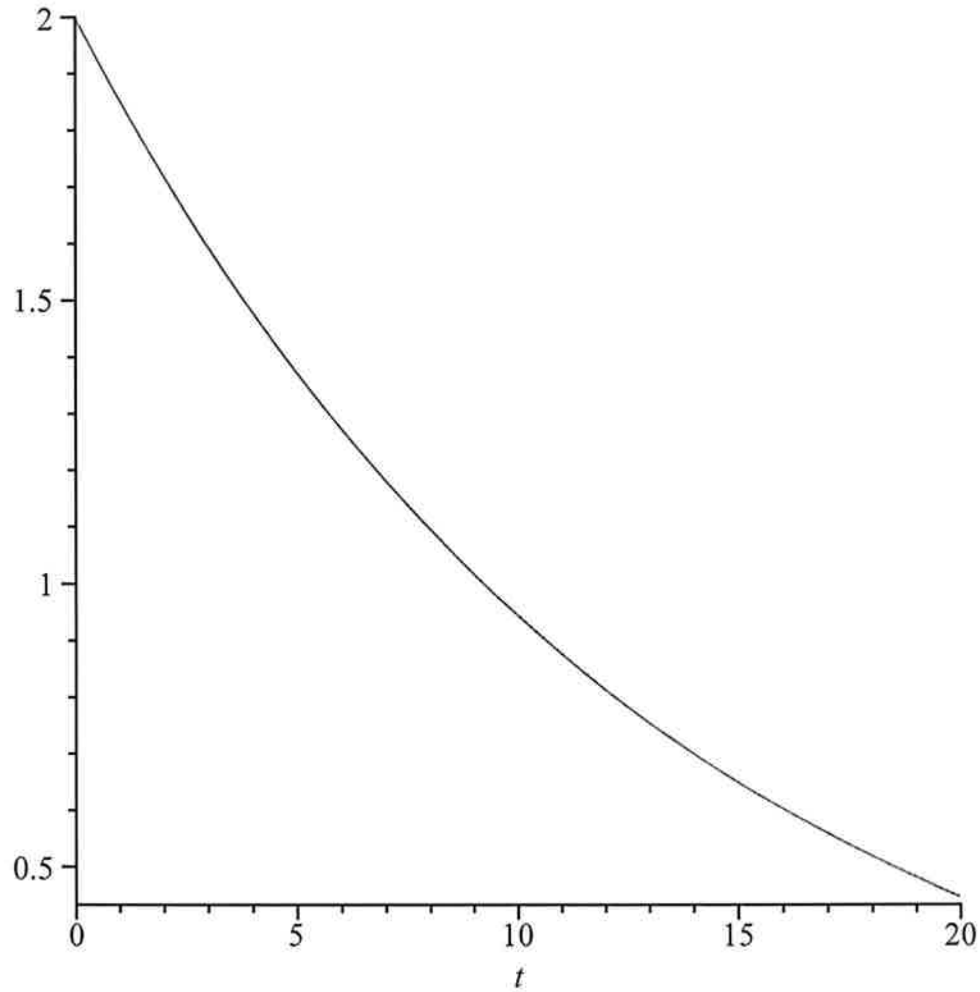
$$x(t) = \_C1\, e^{-\frac{3}{40}t} \qquad \textbf{(1.1.2)}$$

With the initial condition of 2 lbs. of salt in the tank, the solution is:

$x(t) = 2 \cdot \exp\left(-\frac{3}{40} \cdot t\right)$

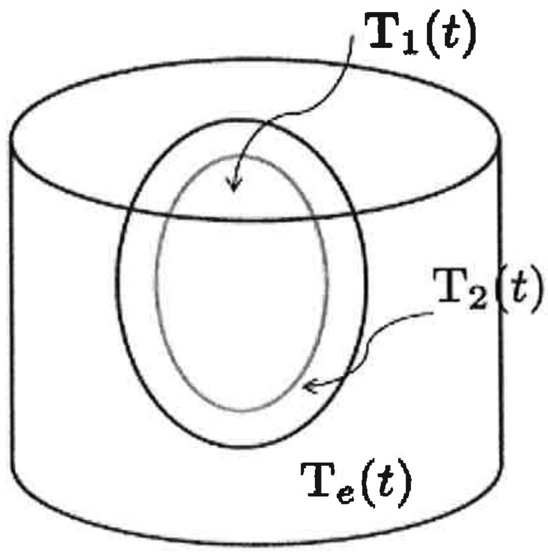
$$x(t) = 2\, e^{-\frac{3}{40}\, t} \qquad \textbf{(1.1.3)}$$

plot(*rhs*(**(1.1.3)**), $t = 0\,..20$)



Transformation into a Second Order Equation

From there, the next logical step in the evolution of the mixing problem is to create a second-order differential equation where the variables in two first-order, coupled in the equations can be solved for and substituted into the other equation to create a second-order equation. An example of this would be:



Consider a refrigerated egg in an ice bath. We can find our D.E.'s from Newton's laws:

$T_1'(t) = -2 \cdot T_1(t) + 2 \cdot T_2(t)$

$$D(T_1)(t) = -2\, T_1(t) + 2\, T_2(t) \quad (1.2.1)$$

$T_2'(t) = 2 \cdot T_1(t) - 5 \cdot T_2(t)$

$$D(T_2)(t) = 2\, T_1(t) - 5\, T_2(t) \quad (1.2.2)$$

We can combine these equations into a second order equation:

$T_1''(t) + 7 \cdot T_1'(t) + 6 \cdot T_1(t) = 0$

$$D^{(2)}(T_1)(t) + 7\, D(T_1)(t) + 6\, T_1(t) = 0 \quad (1.2.3)$$

Where, $T_2(t) = .5 \cdot T_1'(t) + T_1(t)$

dsolve(**(1.2.3)**, $T_1(t)$)

$$T_1(t) = \_C1\, e^{-t} + \_C2\, e^{-6t} \quad (1.2.4)$$

Plugging this into T_2 using initial conditions of $T_1(0) = 40$ and $T_2(0) = 45$:

$T_1 = -10 \cdot \exp(-6 \cdot t) + 50 \cdot \exp(-t)$

$$T_1 = -10\, e^{-6t} + 50\, e^{-t} \quad (1.2.5)$$

$T_2 = 20 \cdot \exp(-6 \cdot t) + 25 \cdot \exp(-t)$

$$T_2 = 20\, e^{-6t} + 25\, e^{-t} \quad (1.2.6)$$

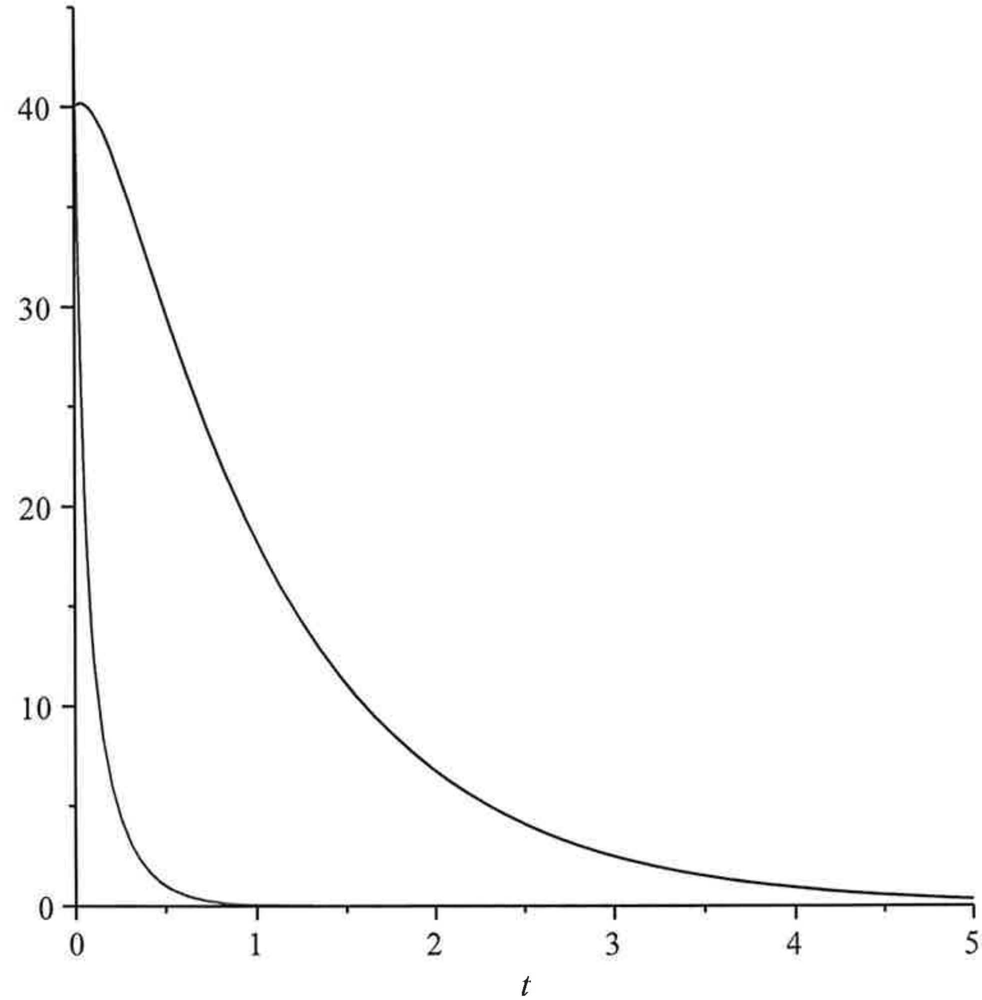
plot1 := *plot*(*rhs*(**(1.2.5)**), *t* = 0 ..5, *color* = *black*)

PLOT(...) **(1.2.7)**

plot2 := *plot*(*rhs*(**(1.2.6)**), *t* = 0 ..5, *color* = *blue*)

PLOT(...) **(1.2.8)**

display(*plot1*, *plot2*)

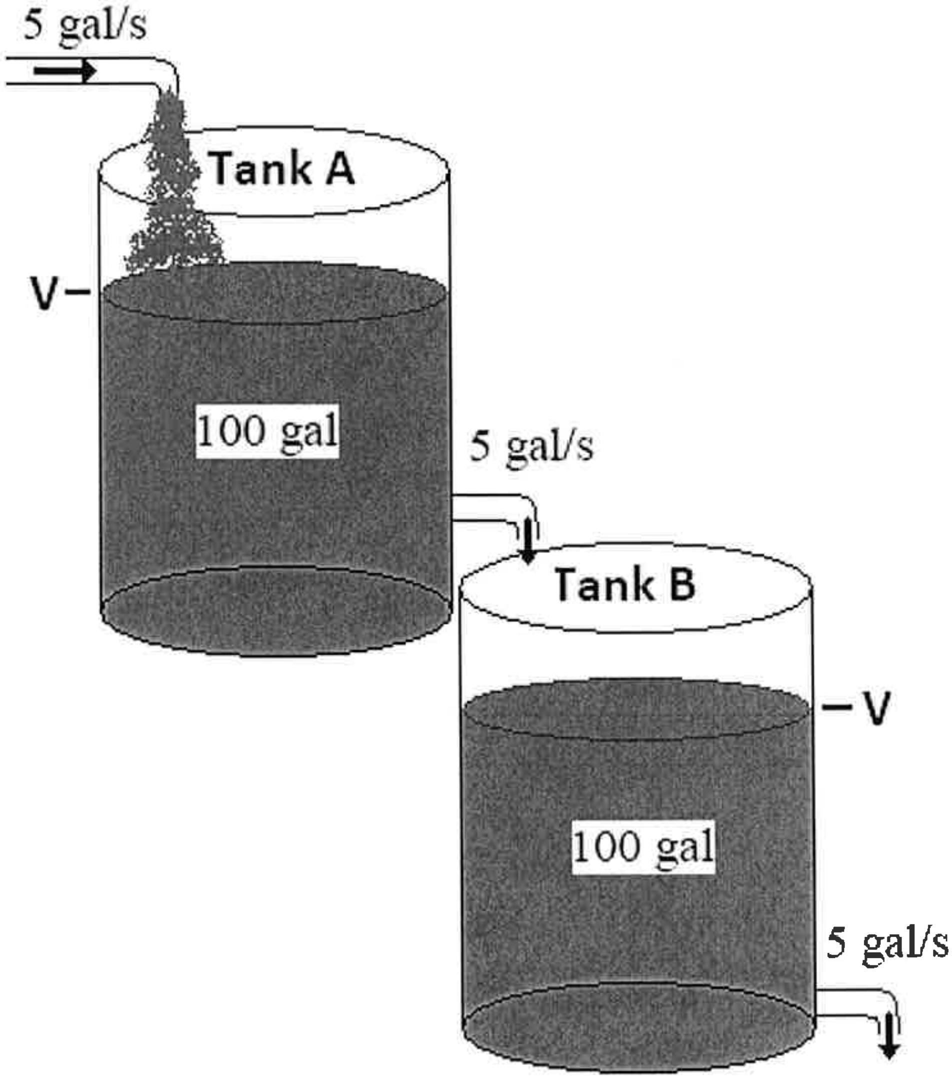


This describes a refrigerated egg being thrown into an ice bath.

System of First Order Equations

The easiest way to solve a second-order mixing problem is to separate the second-order equation into a system of first order equations. This allows for easier solution techniques that can describe the change in concentration and volume of a mixing tank or multiple tanks. An example of a first-order system of equations as applied to a mixing problem is:

Consider now a two tank mixing problem. Both tanks always have a constant volume of 100 gal. Pure water enters Tank 1 at 5 gal/s and salt solution exit at 5 gal/s and enters into Tank 2. Then, salt solution drains from Tank 2 at 5 gal/s. Intially, Tank 1 contains 20 lbs. of salt, and Tank 2 contains 40 lbs. Thus, this yields:



$SYS := \left\{x'(t) = -\frac{x(t)}{20}, y'(t) = \frac{x(t)}{20} - \frac{y(t)}{40}\right\}$

$$\left\{D(x)(t) = -\frac{1}{20}\,x(t), D(y)(t) = \frac{1}{20}\,x(t) - \frac{1}{40}\,y(t)\right\} \quad \textbf{(1.3.1)}$$

$x(t) = 20 \cdot e^{-\frac{1}{20}t}$

$$x(t) = 20\,e^{-\frac{1}{20}t} \quad \textbf{(1.3.2)}$$

$y(t) = 20 \cdot e^{-\frac{1}{20}t} + e^{-\frac{1}{40}t} \cdot 20$

$$y(t) = 20\,e^{-\frac{1}{20}t} + 20\,e^{-\frac{1}{40}t} \quad \textbf{(1.3.3)}$$

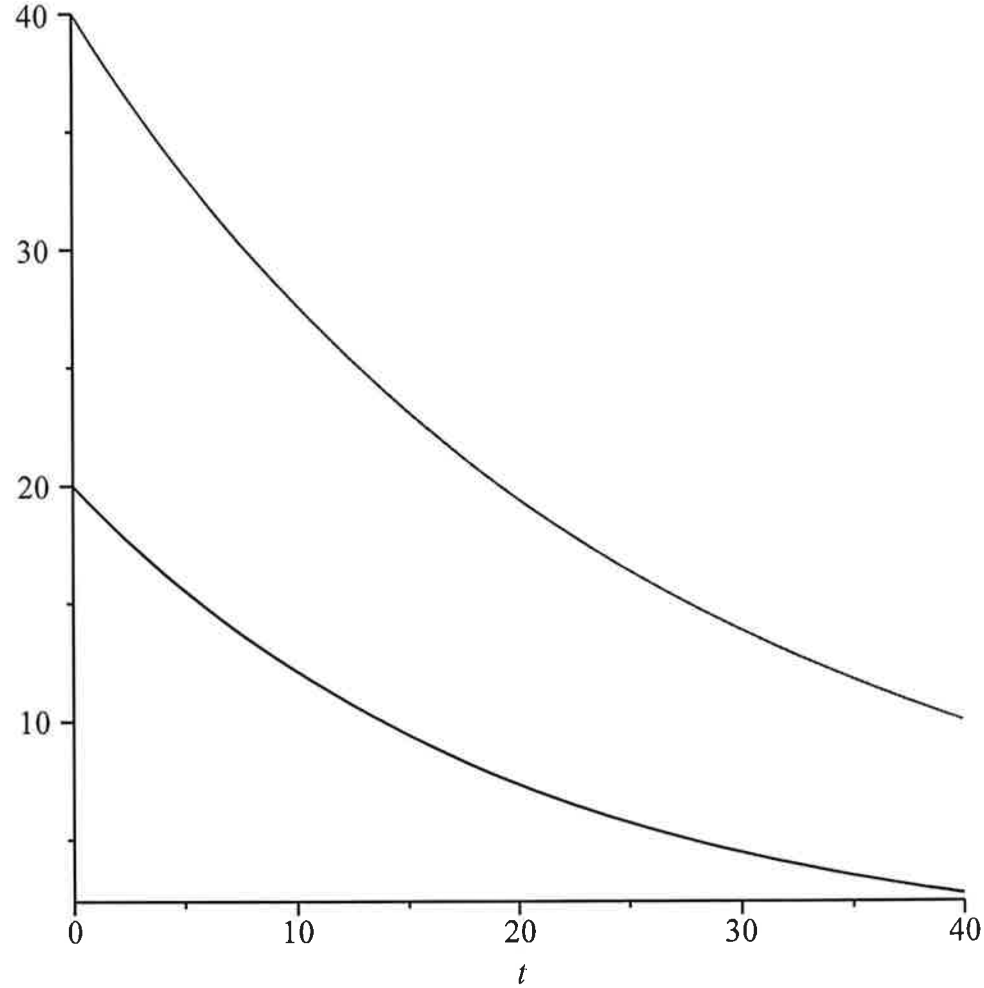
plotx := *plot*(*rhs*(**(1.3.2)**), *t* = 0 ..40, *color* = *black*)

PLOT(...) **(1.3.4)**

ploty := *plot*(*rhs*(**(1.3.3)**), *t* = 0 ..40, *color* = *blue*)

PLOT(...) **(1.3.5)**

display(plotx, ploty)



This shows the salt concentration of Tank 1 approaches 0 much faster than that of Tank 2.

Section 2: Simple Problems with which to Start

Initial Pollution Problem

For a simple example, we are assuming that there is an initial concentration of pollution in Lake Huron is $1000 \frac{gal}{mi^3}$, and the volume remains constant. Lake Erie and all the inputs to the lakes have pure water flow of the rates given in Figure 1. The concentration of pollution in Lake Huron is labeled as the variable x, while the concentration of pollution in Lake Erie is labeled as the variable y.

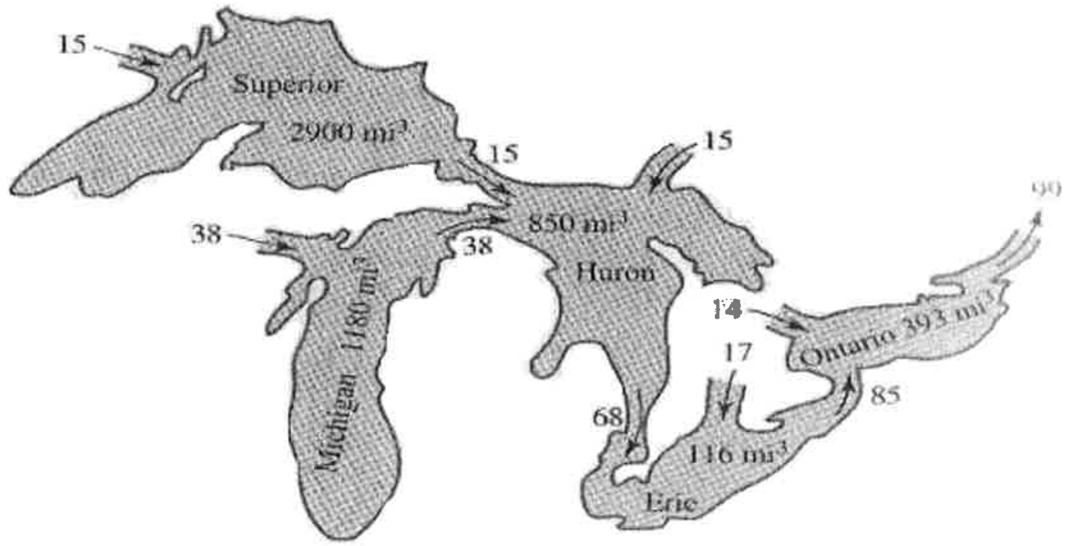


Figure 1

$$x'(t) =- \frac{68}{850} \cdot x(t)$$

$$D(x)(t) = -\frac{2}{25}\, x(t) \tag{2.1.1}$$

$$y'(t) = \frac{68}{850} \cdot x(t) - \frac{85}{116} \cdot y(t)$$

$$D(y)(t) = \frac{2}{25}\, x(t) - \frac{85}{116}\, y(t) \tag{2.1.2}$$

dsolve(**(2.1.1)**, $x(t)$)

$$x(t) = e^{-\frac{2}{25} t}\ \_C1 \tag{2.1.3}$$

So with the intial condition of $\frac{1000\ gal}{mi^3}$, the rate of change of the concentration of the pollution in Lake Huron becomes:

$$x(t) = 1000 \cdot \exp\left(-\frac{2}{25} t\right)$$

$$x(t) = 1000\, e^{-\frac{2}{25} t} \tag{2.1.4}$$

$$subs\left(x(t) = 1000 \cdot \exp\left(-\frac{2}{25} t\right), \textbf{(2.1.2)}\right)$$

$$D(y)(t) = 80\, e^{-\frac{2}{25} t} - \frac{85}{116}\, y(t) \tag{2.1.5}$$

dsolve(**(2.1.5)**, $y(t)$)

$$y(t) = \frac{232000}{1893}\, e^{-\frac{2}{25} t} + e^{-\frac{85}{116} t}\ \_C1 \tag{2.1.6}$$

Solving for the constant term with t=0 and y(t)=0, the equation becomes:

$$y(t) = \frac{232000}{1893} \cdot \exp\left(-\frac{2}{25} t\right) - \frac{232000}{1893} \cdot \exp\left(-\frac{85}{116} t\right)$$

$$y(t) = \frac{232000}{1893}\,\mathrm{e}^{-\frac{2}{25}t} - \frac{232000}{1893}\,\mathrm{e}^{-\frac{85}{116}t} \quad \textbf{(2.1.7)}$$

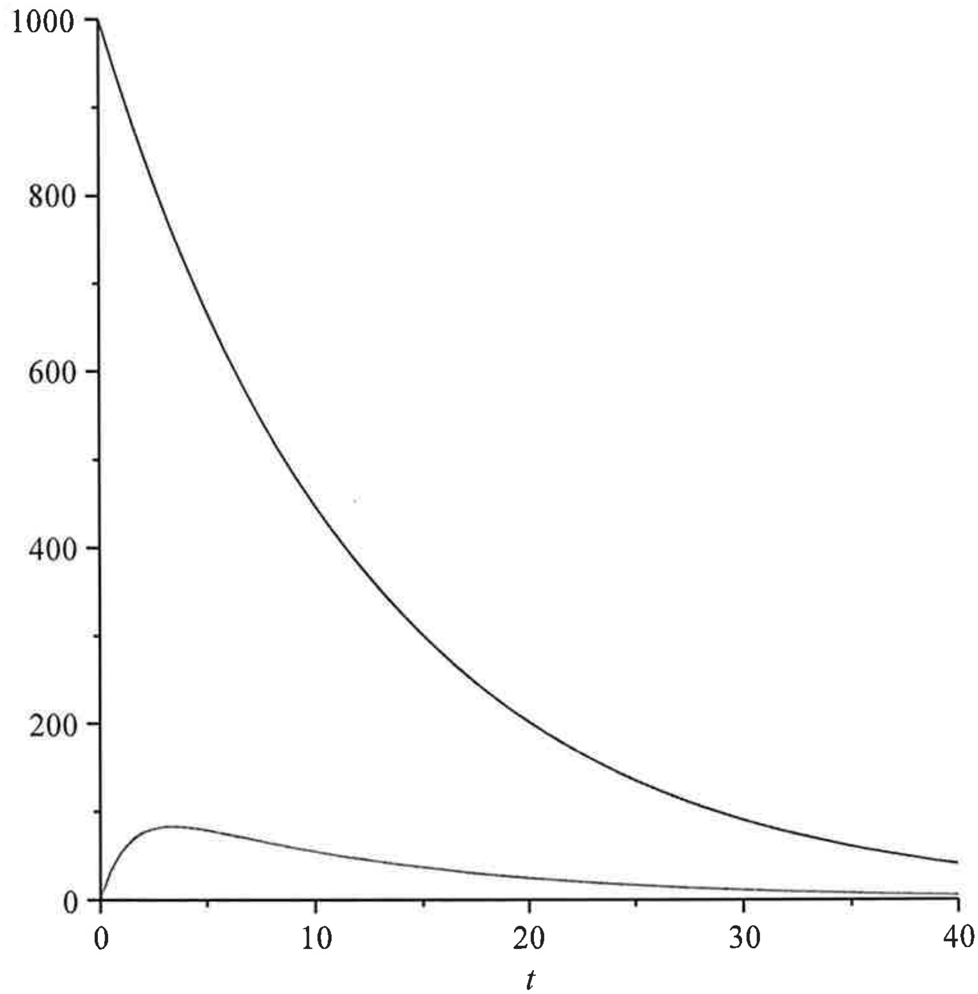
plot1 := *plot*(*rhs*(**(2.1.4)**), *t*=0 ..40)

PLOT(...) **(2.1.8)**

plot2 := *plot*(*rhs*(**(2.1.7)**), *t*=0 ..40.)

PLOT(...) **(2.1.9)**

display(*plot1*, *plot2*)



The blue line represents the amount of pollution in Lake Huron, while the red line represents the pollution in Lake Erie. This says that the pollution in Huron decreases exponentially while the concentration in Erie initially rises then decreases with simultaneous exponential growth and decay.

Static Pollution Inflows

In the next example, consider all five lakes. The inflow rates for both Lake Superior and Lake Michigan stay constant at 15

$\frac{mi^3}{yr}$ and 38 $\frac{mi^3}{yr}$, respectively, where 1 $\frac{mi^3}{yr}$ of pollution makes up part of these inflow rates in each of the lakes. The concentrations for Lakes Superior, Michigan, Huron, Erie, and Ontario are s, m, x, y, and o, respectively.

$$sys := \left\{s'(t) = 1 - \frac{15}{2900}s(t), m'(t) = 1 - \frac{38}{1180}m(t), x'(t) = \frac{15}{2900}s(t) + \frac{38}{1180}m(t) - \frac{68}{850}x(t), y'(t) = \frac{68}{850}x(t) - \frac{85}{116}y(t), o'(t) = \frac{85}{116}y(t) - \frac{99}{393}o(t)\right\}$$

$$\left\{D(m)(t) = 1 - \frac{19}{590}m(t), D(o)(t) = \frac{85}{116}y(t) - \frac{33}{131}o(t), D(s)(t) = 1 - \frac{3}{580}s(t), D(x)(t) = \frac{3}{580}s(t) + \frac{19}{590}m(t) - \frac{2}{25}x(t), D(y)(t) = \frac{2}{25}x(t) - \frac{85}{116}y(t)\right\} \quad (2.2.1)$$

$soln := dsolve(sys);$

$$\left\{m(t) = \frac{590}{19} + e^{-\frac{19}{590}t}\_C5, o(t) = \frac{262}{33} + \frac{6458300}{286122963}e^{-\frac{3}{580}t}\_C4 + \frac{112436300}{438160743}e^{-\frac{19}{590}t}\_C5 + \frac{556750}{1065759}\_C3\, e^{-\frac{2}{25}t} - \frac{11135}{7307}\_C2\, e^{-\frac{85}{116}t} + e^{-\frac{33}{131}t}\_C1, s(t) = \frac{580}{3} + e^{-\frac{3}{580}t}\_C4, x(t) = 25 + \frac{15}{217}e^{-\frac{3}{580}t}\_C4 + \frac{95}{141}e^{-\frac{19}{590}t}\_C5 + \_C3\, e^{-\frac{2}{25}t}, y(t) = \frac{232}{85} + \frac{348}{45787}e^{-\frac{3}{580}t}\_C4 + \frac{260072}{3380193}e^{-\frac{19}{590}t}\_C5 + \frac{232}{1893}\_C3\, e^{-\frac{2}{25}t} + \_C2\, e^{-\frac{85}{116}t}\right\} \quad (2.2.2)$$

Solving for the initial conditions, the equations become:

$$m(t) = \frac{590}{19} - \frac{590}{19}e^{-\frac{19}{590}t}$$

$$m(t) = \frac{590}{19} - \frac{590}{19}e^{-\frac{19}{590}t} \quad (2.2.3)$$

$$s(t) = \frac{580}{3} - \frac{580}{3}e^{-\frac{3}{580}t}$$

$$s(t) = \frac{580}{3} - \frac{580}{3}e^{-\frac{3}{580}t} \quad (2.2.4)$$

$$x(t) = 25 + \frac{15}{217} \cdot \frac{-580}{3}e^{-\frac{3}{580}t} - \frac{590}{19} \cdot \frac{95}{141}e^{-\frac{19}{590}t} + \frac{284125}{30597} \cdot e^{-\frac{2}{25}t}$$

$$x(t) = 25 - \frac{2900}{217}\,\mathrm{e}^{-\frac{3}{580}t} - \frac{2950}{141}\,\mathrm{e}^{-\frac{19}{590}t} + \frac{284125}{30597}\,\mathrm{e}^{-\frac{2}{25}t} \qquad \textbf{(2.2.5)}$$

$$y(t) = \frac{232}{85} - \frac{580}{3}\cdot\frac{348}{45787}\,\mathrm{e}^{-\frac{3}{580}t} - \frac{590}{19}\cdot\frac{260072}{3380193}\,\mathrm{e}^{-\frac{19}{590}t} + \frac{284125}{30597}\cdot\frac{232}{1893}\,\mathrm{e}^{-\frac{2}{25}t} - \frac{7225374128}{813906244215}\,\mathrm{e}^{-\frac{85}{116}t}$$

$$y(t) = \frac{232}{85} - \frac{67280}{45787}\,\mathrm{e}^{-\frac{3}{580}t} - \frac{8075920}{3380193}\,\mathrm{e}^{-\frac{19}{590}t} + \frac{65917000}{57920121}\,\mathrm{e}^{-\frac{2}{25}t} - \frac{7225374128}{813906244215}\,\mathrm{e}^{-\frac{85}{116}t} \qquad \textbf{(2.2.6)}$$

$$o(t) = \frac{262}{33} + \frac{6458300}{286122963}\cdot\mathrm{e}^{-\frac{3}{580}\cdot t}\cdot\frac{-580}{3} + \frac{112436300}{438160743}\cdot\mathrm{e}^{-\frac{19}{590}\cdot t}\cdot\frac{-590}{19} + \frac{556750}{1065759}\cdot\frac{284125}{30597}\cdot\mathrm{e}^{-\frac{2}{25}\cdot t} - \frac{11135}{7307}\cdot\frac{-7225374128}{813906244215}\cdot\mathrm{e}^{-\frac{85}{116}\cdot t} + \mathrm{e}^{-\frac{33}{131}\cdot t}\cdot\frac{-6794419457387840}{14405728606408557}$$

$$o(t) = \frac{262}{33} - \frac{3745814000}{858368889}\,\mathrm{e}^{-\frac{3}{580}t} - \frac{3491443000}{438160743}\,\mathrm{e}^{-\frac{19}{590}t} + \frac{158186593750}{32609028123}\,\mathrm{e}^{-\frac{2}{25}t} + \frac{7225374128}{534100846563}\,\mathrm{e}^{-\frac{85}{116}t} - \frac{6794419457387840}{14405728606408557}\,\mathrm{e}^{-\frac{33}{131}t} \qquad \textbf{(2.2.7)}$$

plot1 := *plot*(*rhs*((**2.2.3**)), *t* = 0 ..1000, *color* = *blue*)

PLOT(...) **(2.2.8)**

plot2 := *plot*(*rhs*((**2.2.4**)), *t* = 0 ..1000, *color* = *black*)

PLOT(...) **(2.2.9)**

plot3 := *plot*(*rhs*((**2.2.5**)), *t* = 0 ..1000, *color* = *red*)

PLOT(...) **(2.2.10)**

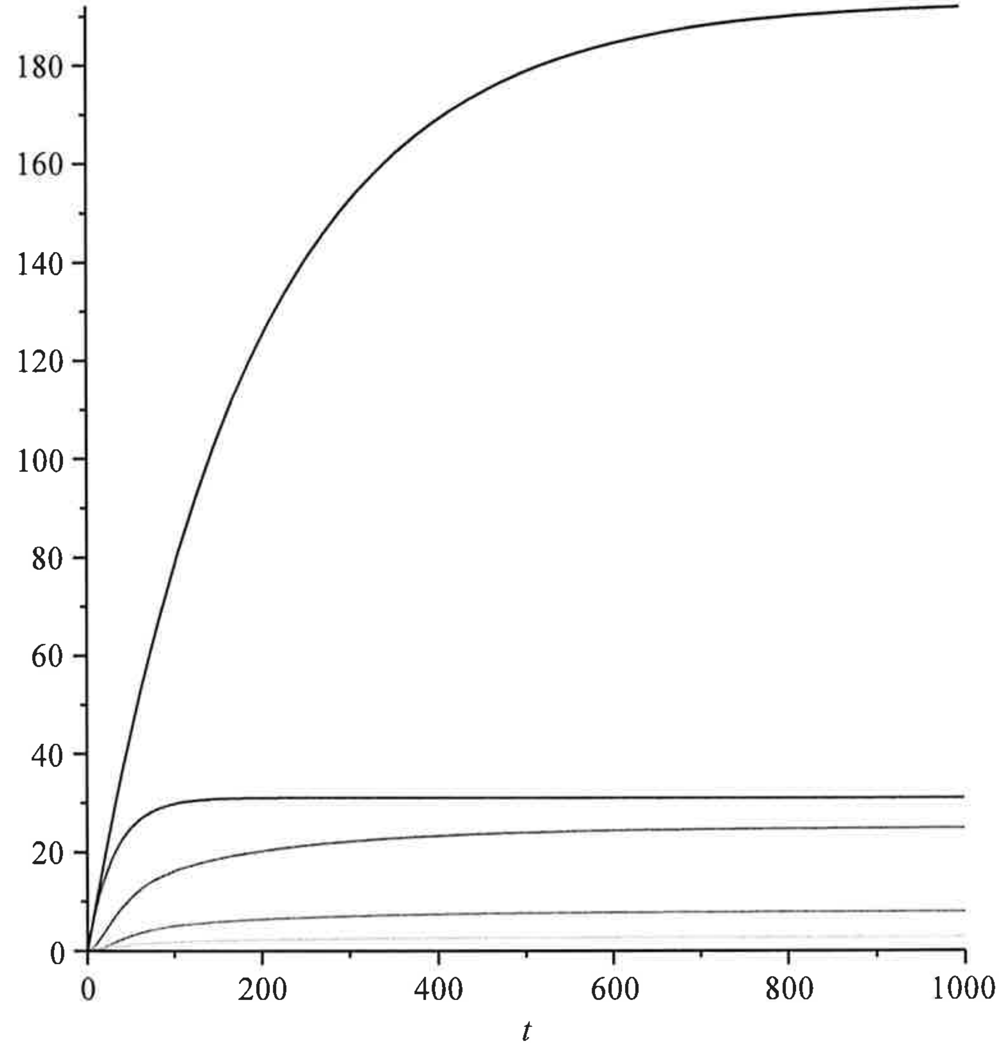
plot4 := *plot*(*rhs*((**2.2.6**)), *t* = 0 ..1000, *color* = *green*)

PLOT(...) **(2.2.11)**

plot5 := *plot*(*rhs*((**2.2.7**)), *t* = 0 ..1000, *color* = *gold*)

PLOT(...) **(2.2.12)**

display(*plot1*, *plot2*, *plot3*, *plot4*, *plot5*)



The black line represents the concentration of Lake Superior, the blue for Lake Michigan, the red for Lake Huron, the green for Lake Erie, and the gold for Lake Ontario. The concentration of the pollutant in Lake Superior is greater than in Lake Michigan because the volume of inflow is greater in Lake Michigan so the pollutant is less concentrated overall. The concentration in Lake Erie is interesting to note as lower than the concentration of Lake Ontario because there is a significant percent of pure water entering as compared to the mixture exiting into Lake Ontario.

Section 3: Extending the Model

Killing the Fishies

In this extension, we are assuming that there is an initial concentration of pollution in Lake Huron is $\frac{.5}{850}$, with no pollution in Lake Erie, and the volume remains constant. Lake Erie's inflow is polluted with a concentration of $\frac{.25\ mi^3}{yr}$ and the inflows coming from Lakes Superior and Michigan are $\frac{.75\ mi^3}{yr}$ and $\frac{1.5\ mi^3}{yr}$ respectively. See Figure 2. The concentration of pollution in Lake Huron is labeled as the variable x, while the concentration of pollution in Lake Erie is labeled as the variable y. After 30 years, environmental agencies introduce a chemical cleaning agent to help remove the pollutant from the water, but they cannot make the factories stop dumping pollution into the lakes. The agencies watch for the pollution levels to drop until fish species are able to survive when the pollution is below 2%. When can PETA reintroduce fish populations into the lakes and have them survive?

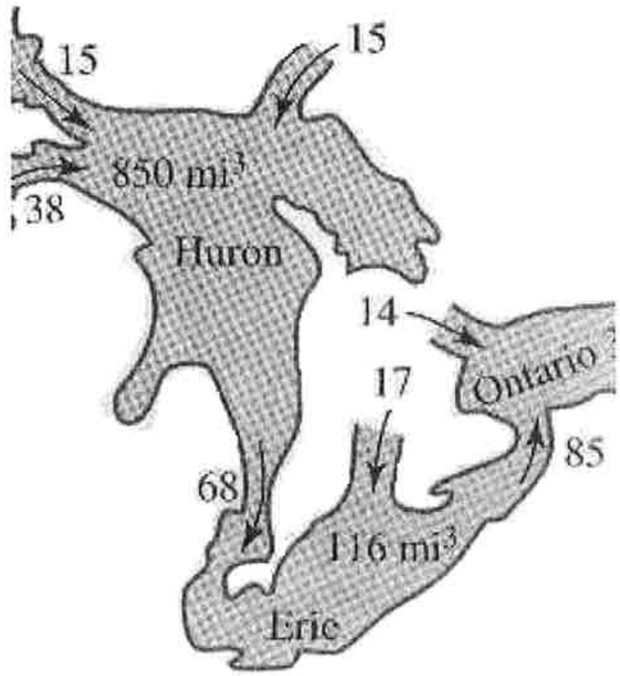


Figure 2.

$$x'(t) = .75 + 1.5 - \frac{68}{850} \cdot x(t)$$

$$D(x)(t) = 2.25 - \frac{2}{25}\, x(t) \quad \textbf{(3.1.1)}$$

$$y'(t) = \frac{68}{850} \cdot x(t) + .25 - \frac{85}{116} y(t)$$

$$D(y)(t) = \frac{2}{25}\, x(t) + 0.25 - \frac{85}{116}\, y(t) \quad \textbf{(3.1.2)}$$

dsolve((**3.1.1**), $x(t)$)

$$x(t) = \frac{225}{8} + e^{-\frac{2}{25} t}\, \_C1 \quad \textbf{(3.1.3)}$$

So with the initial condition, x(t) becomes:

$$x(t) = \frac{225}{8} + \left(.5 - \frac{225}{8}\right) \cdot e^{-\frac{2}{25} t}$$

$$x(t) = \frac{225}{8} - 27.62500000\, e^{-\frac{2}{25} t} \quad \textbf{(3.1.4)}$$

subs((3.1.4), (3.1.2))

$$D(y)(t) = 2.500000000 - 2.210000000\, e^{-\frac{2}{25}t} - \frac{85}{116}\, y(t) \qquad \textbf{(3.1.5)}$$

dsolve((3.1.5), $y(t)$)

$$y(t) = \frac{58}{17} - \frac{6409}{1893}\, e^{-\frac{2}{25}t} + e^{-\frac{85}{116}t}\, \_C1 \qquad \textbf{(3.1.6)}$$

So with the initial conditions, this equation becomes:

$$solve\left(0 = \frac{58}{17} - \frac{6409}{1893}\, e^{-\frac{2}{25}(0)} + e^{-\frac{85}{116}(0)}\, \_C1, \_C1\right)$$

$$-\frac{841}{32181} \qquad \textbf{(3.1.7)}$$

$$y(t) = \frac{58}{17} - \frac{6409}{1893}\, e^{-\frac{2}{25}t} - \frac{841}{32181} \cdot e^{-\frac{85}{116}t}$$

$$y(t) = \frac{58}{17} - \frac{6409}{1893}\, e^{-\frac{2}{25}t} - \frac{841}{32181}\, e^{-\frac{85}{116}t} \qquad \textbf{(3.1.8)}$$

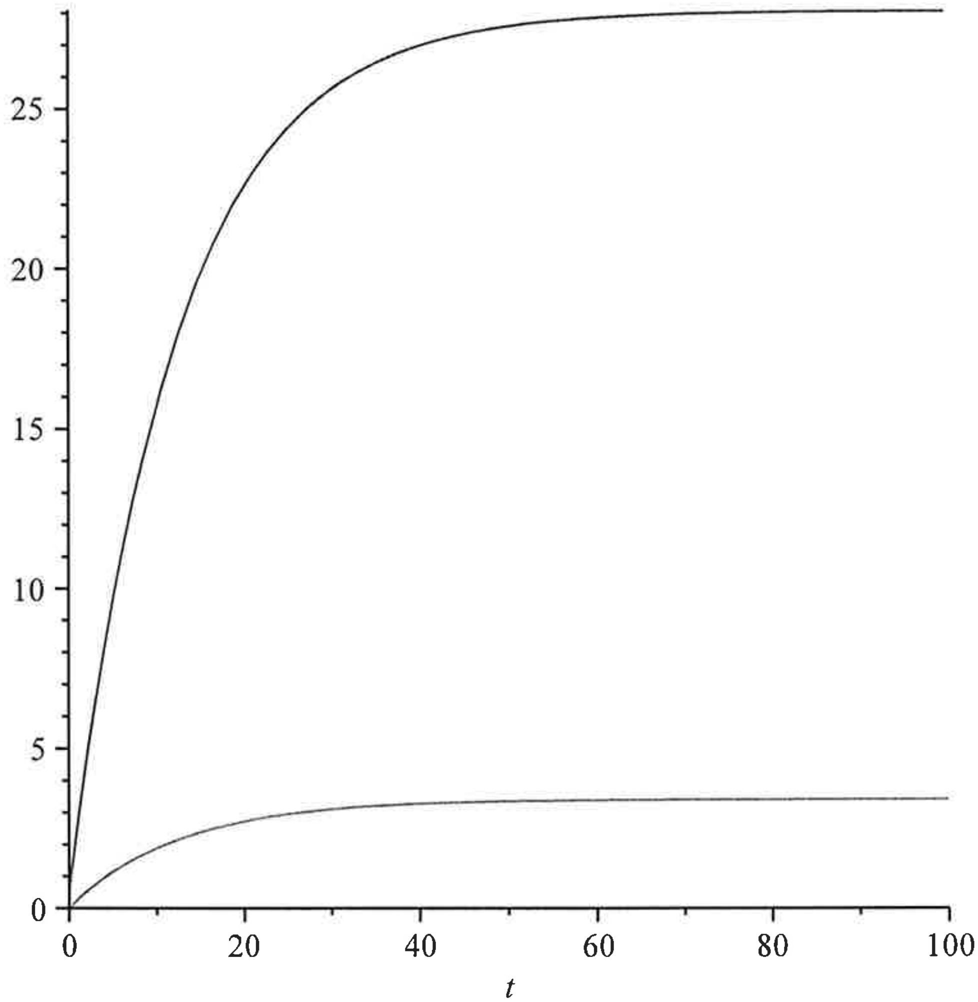
plot1 := *plot*(*rhs*((3.1.4)), $t = 0..100$, *color* = *blue*)

PLOT(...) **(3.1.9)**

plot2 := *plot*(*rhs*((3.1.8)), $t = 0..100$)

PLOT(...) **(3.1.10)**

display(*plot1*, *plot2*)



After 30 years, the fish in Huron start to die because the concentration of the pollutants is:

$$simplify\left(x(30) = \frac{225}{8} - 27.62500000\, e^{-\frac{2}{25}(30)}\right)$$

$$x(30) = 25.61891655 \quad \textbf{(3.1.11)}$$

$$simplify\left(y(30) = \frac{58.0}{17} - \frac{6409.0}{1893} \cdot e^{-\frac{2.0}{25}\cdot(30)} - \frac{841.0}{32181} \cdot e^{-\frac{85.0}{116}\cdot(30)}\right)$$

$$y(30) = 3.104627166 \quad \textbf{(3.1.12)}$$

Furthermore, we see that after approximately 11 years the water becomes toxic to the fish in Lake Huron, while in Lake Erie it takes approximately 20 years to reach this toxic level.

Saving the Fishies

After 30 years, the environmental agencies introduce a chemical cleaning agent, at a rate that will neutralize the pollutant at a rate of $\frac{5\ mi^3}{yr}$, into Lake Huron. Clean water then enters Lake Erie at a rate of $\frac{68\ mi^3}{yr}$. The equations that describe the pollution concentration change become:

$x'(t) = .75 + 1.5 - \frac{68}{850} \cdot x(t) - 5$

$$D(x)(t) = -2.75 - \frac{2}{25}\, x(t) \qquad (3.2.1)$$

$25.61891655 + \frac{275}{8}$

$$59.99391655 \qquad (3.2.2)$$

dsolve((**3.2.1**), $x(t)$)

$$x(t) = -\frac{275}{8} + e^{-\frac{2}{25}t}\, \_C1 \qquad (3.2.3)$$

With the initial conditions, the equation for Lake Huron becomes:

$x(t) = -\frac{275}{8} + 59.99391655 \cdot e^{-\frac{2}{25}t}$

$$x(t) = -\frac{275}{8} + 59.99391655\, e^{-\frac{2}{25}t} \qquad (3.2.4)$$

subs$\left((\mathbf{3.2.4}), y'(t) = \frac{68}{850} \cdot x(t) + .25 - \frac{85}{116} y(t)\right)$

$$D(y)(t) = -2.500000000 + 4.799513324\, e^{-\frac{2}{25}t} - \frac{85}{116}\, y(t) \qquad (3.2.5)$$

dsolve((**3.2.5**), $y(t)$)

$$y(t) = -\frac{58}{17} + \frac{34796471599}{4732500000}\, e^{-\frac{2}{25}t} + e^{-\frac{85}{116}t}\, \_C1 \qquad (3.2.6)$$

$3.104627166 - \left(-\frac{58}{17} + \frac{34796471599}{4732500000}\right)$

$$-0.836269850 \qquad (3.2.7)$$

$y(t) = -\frac{58}{17} + \frac{34796471599}{4732500000} \cdot e^{-\frac{2}{25}t} - 0.836269850 \cdot e^{-\frac{85}{116}t}$

$$y(t) = -\frac{58}{17} + \frac{34796471599}{4732500000}\, e^{-\frac{2}{25}t} - 0.836269850\, e^{-\frac{85}{116}t} \qquad (3.2.8)$$

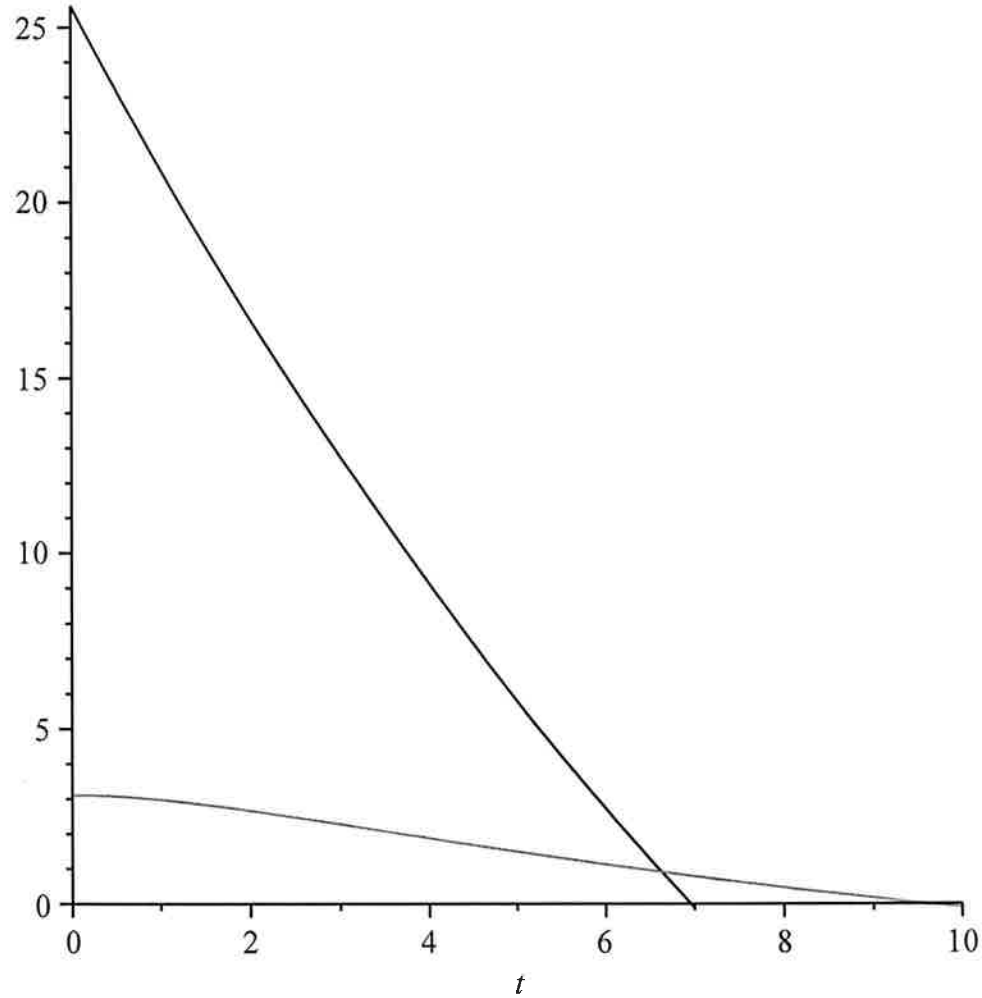
plot3 := *plot*(*rhs*((**3.2.4**)), *t* = 0 ..7, *color* = *black*)

$$PLOT(\ldots) \qquad (3.2.9)$$

plot4 := *plot*(*rhs*((**3.2.8**)), *t* = 0 ..10)

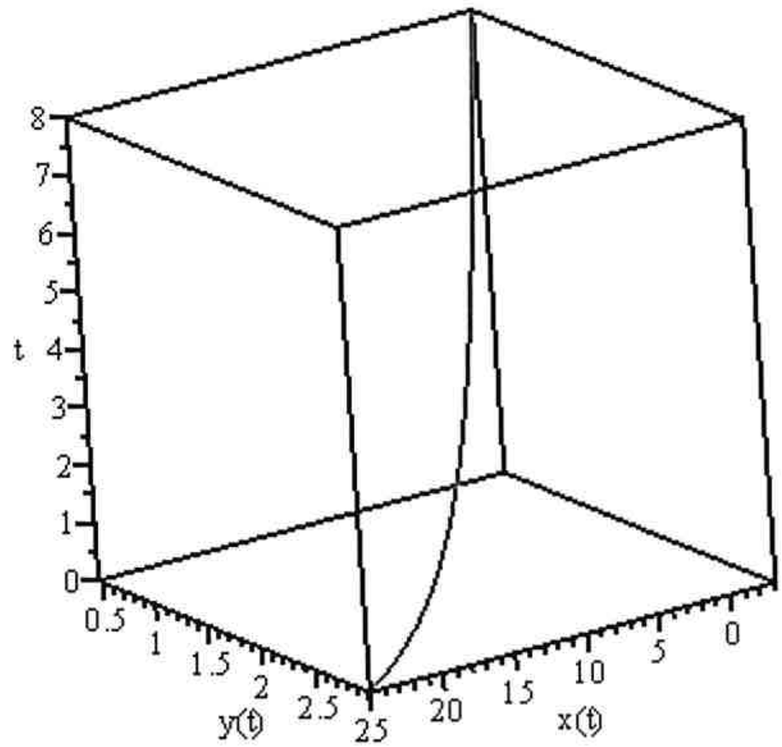
$$PLOT(\ldots) \qquad (3.2.10)$$

display(*plot3*, *plot4*)



When you first introduce the cleaning agent, the pollution in Lake Erie from the factories decreases, and the chemical agents begin to clean Lake Erie as well. Lake Huron (black) is completely clean from pollutants after 7 years of the cleaning agent introduction, and Lake Erie (red) is clean after 10 years. Fish can be reintroduced when the pollutants in Lake Huron reaches $17\ \frac{gal}{mi^3}$, which is around three years after the introduction of the chemical cleaners.

DEplot3d([**(3.2.1)**, **(3.2.5)**], $[x(t), y(t)]$, $t=0..8$, $[[x(0)=25, y(0)=3]]$, *stepsize* = 0.05, *scene* = $[x(t), y(t), t]$, *linecolor* = *blue*, *thickness* = 2);



This plot shows that Lake Huron's concentration changes faster initially after the introduction of the cleaning chemical, but over time the relative rates of change of both lakes begin to level off.