Arclength And Surface Area Revolutions
Section 6.4

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The Idea of Arc Length

The idea of calculating the length along a curve represents the basic idea of calculus: deal with a hard problem by breaking it up into small problems (actually infinitesimal) and obtain the answer in the limit. To calculate the arc length of a curve we divide the curve defined on \([a, b]\) into segments \(s_1, \ldots, s_n\) of equal length \(\Delta s\). The distance \(\Delta s\) can be approximated by the distances \(\Delta x\) and \(\Delta y\) using the Pythagorean theorem as shown in the figure below.

\[
(\Delta s)^2 \approx (\Delta x)^2 + (\Delta y)^2.
\]

As we let \(\Delta x \to 0\), this approximation becomes better as the curve actually approaches becomes more linear, and we obtain

\[
(ds)^2 = (dx)^2 + (dy)^2.
\]
A Better Form for Arc Length

In our notation we have

$$(ds)^2 = (dx)^2 + (dy)^2 \implies ds = \sqrt{dx^2 + dy^2}.$$  

A variant of this formula can be obtained by factoring out the $dx$. In this case we obtain,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.$$  

Integrating both sides of this expression gives us

$$\text{Arc Length} = \int ds = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.$$  

An alternative form used by many texts, is to let $y = f(x)$ in which case we have

$$\text{Arc Length} = \int_a^b \sqrt{1 + f'(x)^2} \, dx.$$  

We will use these forms interchangeably.
Circumference of a Circle

Example

Show that the circumference of a circle of radius $a$ is $2\pi a$.

The upper half of a circle of radius $a$ centered at $(0, 0)$ is given by the function $y = \sqrt{a^2 - x^2}$ for $-a \leq x \leq a$. 
Solution continued...

The circle has vertical tangent lines at $x = \pm a$ and $f'(\pm a)$ is undefined, which prevents us from using the arc length formula. So we’ll have to set up the problem to avoid this. Instead we consider one-eighth of the circle on the interval $[0, a/\sqrt{2}]$. (See the figure on the previous slide.) We determine that $f'(x) = -\frac{x}{\sqrt{a^2-x^2}}$. The arc length corresponding to this region is given by

$$\int_0^{a/\sqrt{2}} \sqrt{1 + f'(x)^2} \, dx = \int_0^{a/\sqrt{2}} \sqrt{1 + \left(-\frac{x}{\sqrt{a^2-x^2}}\right)^2} \, dx$$

$$= a \int_0^{a/\sqrt{2}} \frac{dx}{\sqrt{a^2 - x^2}}$$

$$= a \sin^{-1} \left( \frac{x}{a} \right) \bigg|_0^{a/\sqrt{2}}$$

$$= \frac{\pi a}{4}$$

It follows that the circumference of the full circle is $8(\pi a/4) = 2\pi a$ units.
Arc Length in Polar Coordinates

Observe in the previous calculation we used the term units, when in reality the units are actually radians. Note that if \( a = 1 \), then our answer for the eighth of the circle’s arc length is \( \pi/4 \), but \( \theta = \pi/4 \)! Recall, the definition of the radian as presented in most trigonometry courses is the unit of the angle, equal to the central angle whose arc is equal in length to the radius, and is given formula \( s = r\theta \). This suggests that maybe this problem is more appropriately described in terms of polar coordinates.

Recall that \( x = r \cos \theta \) and \( y = r \sin \theta \) are the conversion formulas. This means that

\[
dx = \frac{dr}{d\theta} \cos \theta - r \sin \theta \quad \text{and} \quad dy = \frac{dr}{d\theta} \sin \theta + r \cos \theta.\]

Calculating \( ds \) we get

\[
ds^2 = \left( \left( \frac{dx}{d\theta} \right)^2 + \left( \frac{dy}{d\theta} \right)^2 \right) d\theta
\]

\[
= \left( \left( \frac{dr}{d\theta} \cos \theta - r \sin \theta \right)^2 + \left( \frac{dr}{d\theta} \sin \theta + r \cos \theta \right)^2 \right) d\theta
\]

\[
= \left( r^2 + \left( \frac{dr}{d\theta} \right)^2 \right) d\theta
\]
Solution via Polar Coordinates

Our previous calculation gives the arc length formula in polar coordinates

\[ ds = \sqrt{r^2 + \left( \frac{dr}{d\theta} \right)^2} \, d\theta. \]

For a circle of radius \( a \), the formula is given by \( r(\theta) = a \), where \( 0 \leq \theta \leq 2\pi \). Using the arc length formula we have

\[
\int ds = \int_0^{2\pi} \sqrt{a^2 + 0} \, d\theta \\
= \int_0^{2\pi} a \, d\theta \\
= 2\pi a
\]

This is consistent with the formula for the circumference of a circle, and was a much easier integration as anticipated.
A More Difficult Example

Example

Find the length of the curve $y = \ln(x + \sqrt{x^2 - 1})$ on the interval $[1, \sqrt{2}]$. 

\[ y = \ln(x + \sqrt{x^2 - 1}) \]
\[ x = \frac{e^y + e^{-y}}{2} \]
Solution continued...

This graph has a vertical tangent at $x = 1$, so we will have to adjust our strategy. The easiest trick to use is to express this function in terms of $y$ (since there are no horizontal asymptotes present). In this case we solve for $y = \ln(x + \sqrt{x^2 - 1})$ in terms of $x$,

$$e^y = x + \sqrt{x^2 - 1}$$
$$e^y - x = \sqrt{x^2 - 1}$$
$$e^{2y} - 2e^y x = -1$$

$$x = \frac{e^{2y} + 1}{2e^y} = \frac{e^y + e^{-y}}{2}$$

The interval $[1, \sqrt{2}]$ corresponds to the $y$-interval $[0, \ln(\sqrt{2} + 1)]$. The arc length is calculated as

$$\int_0^{\ln(\sqrt{2}+1)} \sqrt{1 + \left(\frac{e^y - e^{-y}}{2}\right)^2} \, dy = 1.$$ 

Note that $\frac{1}{2}(e^y + e^{-y})$, the hyperbolic cosine $\cosh(y)$ and its derivative $\sinh(y)$, appeared in this problem.
Made For Parametric Equations

Suppose we have a smooth curve described by the parametric equations $x = f(t)$ and $y = g(t)$. The term *smooth* means $f$ and $g$ have continuous first derivatives that are not simultaneously zero. It is a simple exercise to rewrite our arc length formula in this case. We have

$$
\text{Arc Length} = \int_a^b \sqrt{dx^2 + dy^2}
$$

$$
= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt
$$

$$
= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt.
$$

Example

Recall the formula for the asteroid is given by

$$
x = \cos^3 t, \quad y = \sin^3 t, \quad 0 \leq t \leq 2\pi.
$$

Find the arc length of one trace of the asteroid.
Solution continued...

We use the symmetry and calculate only the arc length of the first quadrant portion. We have

$$\left(\frac{dx}{dt}\right)^2 = 9 \cos^4 t \sin^4 t$$

$$\left(\frac{dy}{dt}\right)^2 = 9 \sin^4 t \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = 3 \cos t \sin t$$

Therefore the length of the first quadrant portion is given by

$$\int_0^{\pi/2} 3 \cos t \sin t \, dt = 3/2.$$ 

This means the total length of the asteroid is $4(3/2) = 6$. 
Surface Areas via Revolution

In a previous lecture, we learned how to find the length of a curve using the arclength integral. This was an important step because it allows us to find the surface area created by rotating a curve about an axis.

If we follow the same strategy we used with arc length, we can approximate the original curve with a piecewise linear function. When a segment of this approximation is rotated about an axis, it creates a simpler figure whose surface area approximates the actual surface area. These bands can be considered a portion of a circular cone, and so we will need to know something about the surface areas of cones.

Example

What is the formula for the lateral surface area of the cone and the truncated cone?
Surface Area of Cones

A circular cone with base radius $r$ and slant height $l$ can be unfolded by cutting along the slant height. The object obtained is a sector of a circle of radius $l$, pictured below.

![Diagram of a cone unfolded into a sector]

We know that in general, the area of a sector a circle such as the one shown above is given by

$$A = \frac{1}{2} l^2 \theta = \frac{1}{2} l^2 \left( \frac{2\pi r}{l} \right) = \pi rl.$$

To derive the formula for the surface area of revolution, we need the truncated cone, shown below:

![Diagram of a truncated cone]
Truncated Cone Surface Area

The area of the band with slant height $l$ and upper and lower radii $r_1$ and $r_2$ is found by subtracting the areas of the two cones in the previous slide:

$$A = \pi r_2(l_1 + l) - \pi r_1 l_1 = \pi [(r_2 - r_1)l_1 + r_2 l].$$

Using similar triangles, we compute

$$\frac{l_1}{r_1} = \frac{l_1 + l}{r_2}$$

which gives

$$r_2 l_1 = r_1 l_1 + r_1 l \iff (r_2 - r_1)l_1 = r_1 l.$$ 

Updating our original equation we have

$$A = \pi (r_1 l + r_2 l) = 2\pi rl$$

where $r$ is the average radius of the band.
The Surface Area Integral

We apply the previous formula to our strategy. Consider the surface shown below which is obtained by rotating the curve $y = f(x), a \leq x \leq b$ about the $x$-axis where $f$ is positive and has a continuous derivative. We divide the interval $[a, b]$ into $n$ subintervals with endpoints $x_0, x_1, \ldots x_n$ and equal width $\Delta x$. If $y_i = f(x_i)$, then a band with length $l$, on the interval $[x_{i-1}, x_i]$ will have radii $y_{i-1}, y_i$. Using the formula for the surface area of the truncated cone from before, we have

$$A = \pi (y_{i-1} + y_i) \cdot l.$$

As $\Delta x \to 0$, $y_{i-1} + y_i \approx 2y_i$, so we have

$$A = 2\pi y_i \sqrt{1 + \left(\frac{dy_i}{dx}\right)^2} \Delta x = 2\pi f(x_i) \sqrt{1 + f'(x_i)^2} \Delta x.$$

The area of the complete surface of revolution is then given by

$$S = \sum_{i=0}^{n} 2\pi f(x_i) \sqrt{1 + f'(x_i)^2} \Delta x \to \int_{a}^{b} 2\pi f(x) \sqrt{1 + [f'(x)]^2} \, dx.$$

This integral (like the arc length integral before) can take on many forms.
**Best Way to Think About This**

These formulas can be remembered by thinking of $2\pi y$ and $2\pi x$ as the circumference of a circle traced out by the point $(x, y)$ on the curve as it is rotated about the $x$-axis or $y$-axis, respectively.

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**Example**

Find the surface area of the solid of revolution obtained by rotating the curve $f(x) = \sqrt{x}$ on the interval $1 \leq x \leq 4$ about the $x$-axis.
Solution...

For \( f(x) = \sqrt{x} \), we have \( dy/dx = 1/2\sqrt{x} \). This gives the surface area

\[
S = \int_{1}^{4} 2\pi \sqrt{x} \sqrt{1 + \frac{1}{4x}} \, dx = \pi \int_{1}^{4} \sqrt{4x + 1} \, dx.
\]

Substituting \( u = 4x + 1 \) and \( du = 4 \, dx \). Changing limits to \( u = 17 \) and \( u = 5 \), we obtain

\[
S = \frac{\pi}{4} \int_{5}^{17} \sqrt{u} \, du = \frac{\pi}{4} \left( \frac{2}{3} u^{3/2} \right)_{5}^{17} = \frac{\pi}{6} (17\sqrt{17} - 5\sqrt{5}).
\]